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A unified theory for construction of arbitrary speeds ($0 \leq v < \infty$) solutions of the relativistic wave equations

WALDYR A. RODRIGUES, Jr.[1] and JOSÉ E. MAIORINO[2]

[1]*Instituto de Matemática, Estatística e Computação Científica*
IMECC-UNICAMP; CP 6065, 13081-970, Campinas, SP, Brazil
[2]*Instituto de Física "Gleb Wataghin"*
IFGW-UNICAMP; CP 6165, 13081-970, Campinas, SP, Brazil

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Abstract—Representing the relativistic physical fields as sections of the Clifford Bundle (or of the Spin-Clifford Bundle) of Minkowski spacetime we show that all the relativistic wave equations satisfied by these fields possess solutions traveling with arbitrary speeds $0 \leq v < \infty$. By giving rigorous mathematical definitions of reference frames and of the Principle of Relativity (PR) we prove that physical realizations of the $v > 1$ solutions of, *e.g.*, the Maxwell equations imply in a breakdown of the PR, but in no contradiction at all with known physical facts.

1. INTRODUCTION

In this paper we present methods for constructing solutions with arbitrary speeds $(0 \leq v < \infty)$[1] of the main relativistic wave equations of physics, namely the (scalar) homogeneous wave equation (HWE) and Maxwell, Weyl, Klein-Gordon and Dirac equations. Several examples are worked in detail.

In Section 2 we show how to represent all fields mentioned above as sections of the Cliford bundle $\mathcal{C}\ell(M)$ or of the Spin-Clifford bundle $\mathcal{C}\ell_{Spin_+(1,3)}(M)$ of Minkowski spacetime [1, 2, 3]. We then show that if the HWE and the Klein-Gordon equation (KGE) have solutions with arbitrary speeds $0 \leq v < \infty$, it then follows that also Maxwell, Weyl and Dirac equations have such solutions.

The solutions we are going to exhibit for the relativistic wave equation are families of *undistorted progressive waves* (UPWs). By UPW, following Courant and Hilbert [4] we mean that the waves are distortion-free, *i.e.* that they are translationally invariant and thus do not spread, or that they reconstruct their original form after a certain period of time.

In Section 3 we present $0 \leq v < \infty$ UPW solutions of the HWE and of the KGE. The meaning of the velocity of propagation of a UPW and the concepts of phase and group velocities are discussed in details. Misconceptions regarding the velocity of propagation of energy of a given wave are clarified. In Section 4 we present $v \neq 1$ UPW solutions of Maxwell equations, discussing the remarkable characteristics of these solutions in contrast to the $v = 1$ solutions. One of the principal novelties is that the $v \neq 1$

[1]We use units such that $c = 1$, where c is the so-called speed of light in vacuum.

solutions have in general non-null field invariants and are not transverse waves. We also discuss in this section the important problem of the velocity of transport of energy for the $v > 1$ UPWs solutions of Maxwell equations.

It must be said that there are experimental data [5, 6] obtained with techniques developed by Lu and Greenleaf [7, 8] showing the existence of pressure waves moving with speeds[2] $v < c_s$ and $v > c_s$ which confirm the theoretical predictions, showing in particular that the energy associated with the pressure waves can travel with speed $v_\varepsilon > c_s$. In [6] it is discussed the possibility of designing physical devices for launching in physical space the $v \neq 1$ solutions of Maxwell equations.

If this is indeed possible we must investigate the status of the Principle of Relativity (PR). This is done in Section 5, where we give a thoughtful and rigorous mathematical definition of this principle and its relation with Lorentz invariance. We prove that the $v > 1$ solutions of Maxwell equations imply necessarily in a breakdown of the PR, which is necessary in order to avoid logical contradictions. In Section 6 we present our conclusions.

2. A UNIFIED THEORY FOR CONSTRUCTION OF UPW SOLUTIONS OF MAXWELL, DIRAC AND WEYL EQUATIONS

To fix the notations we recall here the main results concerning the theory of Clifford algebras (and bundles) and their relationship with the Grassmann algebras (and bundles). In particular a self-consistent presentation of the so called spacetime and Pauli algebras is given. Also the concept of Dirac-Hestenes spinors and their relationship with the usual Dirac spinors used by physicists is clarified. We introduce moreover the concept of the Clifford bundle of spacetime and the Clifford calculus. As we shall see, this formalism provides a unified theory for the construction of UPW subluminal, luminal and superluminal solutions of Maxwell, Dirac and Weyl equations once we have arbitrary speed ($0 \leq v < \infty$) solutions of the homogeneous wave equation (HWE) and of the Klein-Gordon equation KGE. More details on these topics can be found in [1, 2, 3].

2.1. Exterior, Grassmann and Clifford algebras

Let V be a n-dimensional real[3] vector space, V^* its dual space, T^rV the space of r-contravariant tensors over $V(r \geq 0, T^0V \equiv \mathbb{R}, T^1V = V)$ and let TV be the tensor algebra of V.

We recall that the *exterior algebra* of V is the quotient algebra

$$\bigwedge V = TV/J \tag{2.1}$$

where J is the bilateral ideal in TV generated by elements of the form $u \otimes v + v \otimes u, u, v \in V$. The elements of $\bigwedge V$ will be called multivectors, or multiforms if they are elements of $\bigwedge V^*$.

Let $\rho : TV \to \bigwedge V$ be the canonical projection of TV onto $\bigwedge V$. Multiplication in $\bigwedge V$ will be denoted as usual by $\wedge : \bigwedge V \to \bigwedge V$ and called exterior (or wedge or Grassmann) product. We have

$$A \wedge B = \rho(A \otimes B). \tag{2.2}$$

[2] Here c_s is the speed of sound in water.
[3] Here $\mathbb{R}$ denotes the real field.

We recall that $\bigwedge V$ is a 2^n-dimensional associative algebra with unity.[4] In addition it is a Z-graded algebra, *i.e.*,

$$\bigwedge V = \bigoplus_{r=0}^{n} \bigwedge^r V; \quad \bigwedge^r V \wedge \bigwedge^s V \subset \bigwedge^{r+s} V \; ; \tag{2.3}$$

$r, s \geq 0$, where $\bigwedge^r V = \rho(T^r V)$ is the $\binom{n}{r}$- dimensional subspace of r-vectors, $\bigwedge^0 V = I\!R$, $\bigwedge^1 V = V$; $\bigwedge^r V = \{\phi\}$ if $r > m$. If $A \in \bigwedge^r V$ for some $r(r = 0, \ldots, n)$ then A is said to be homogeneous, otherwise it is said to be inhomogeneous. For $A_p \in \bigwedge^p V$ and $B_q \in \bigwedge^q V$ we have

$$A_p \wedge B_q = (-1)^{pq} B_q \wedge A_p. \tag{2.4}$$

Let $(E_1, \ldots, E_n)$ be a basis for V. Then a basis for $\bigwedge V$ is

$$\{1, E_1, \ldots E_n, E_1 \wedge E_2, \ldots, E_1 \wedge E_n, \ldots, E_{n-1} \wedge E_n, E_1 \wedge E_2 \wedge E_3, \ldots, E_1 \wedge E_2 \wedge \ldots \wedge E_n\} \tag{2.5}$$

Then, if $A \in \bigwedge V$, we can write

$$A = S + A^i E_i + \frac{1}{2!} A^{ij} E_i \wedge E_j + \frac{1}{3!} A^{ijk} E_i \wedge E_j \wedge E_k + \ldots + P E_1 \wedge E_2 \wedge \ldots \wedge E_n, \tag{2.6}$$

where $S, A^{ij}, A^{ijk}, \ldots, P \in I\!R$ and $A^{ij} = -A^{ji}$, etc... The element

$$E_{n+1} = E_1 \wedge E_2 \ldots \wedge E_n \tag{2.7}$$

is called the pseudoscalar of the algebra $\bigwedge V$. (The analogous element for $\bigwedge V^*$ is also called the volume element). We define the projector $\langle\rangle_k : \bigwedge V \to \bigwedge^k V$ by $\bigwedge V \ni A \mapsto A_k$, for $A = \sum_{k=0}^{n} A_k, A_k \in \bigwedge^k V$.

Now let $g \in T^2 V^*$ be a metric for V of signature (p, q), *i.e.*, $g : V \times V \to I\!R$ and let $g^{-1} \in T^2 V, g^{-1} e : V^* \times V^* \to I\!R$ be the metric of the dual space. If $u, v \in V$ and $\alpha, \beta \in V^*$ such that $\alpha(u) = 1, \beta(v) = 1$ we have

$$g(u, v) = g^{-1}(\alpha, \beta) \; . \tag{2.8}$$

We can use g to induce a scalar product on $\bigwedge V$, $G : \bigwedge V \times \bigwedge V \to I\!R$. We define

$$G(A, B) = \det(g(u_i, v_j)) \tag{2.9}$$

for homogeneous multivectors $A = u_1 \wedge \ldots \wedge u_r \in \bigwedge^r V$; $B = v_1 \wedge \ldots \wedge v_r \in \bigwedge^r V$, $u_i, v_j \in \bigwedge^1 V$, $i, j = 1, \ldots, r$. This scalar product is extended to all V due to linearity and associativity. $G(A, B) = 0$ if $A \in \bigwedge^r V$, $B \in \bigwedge^s V$, $r \neq s$. When both $A, B \in \bigwedge^0 V$, $G(A, B)$ means the product AB. The algebra $\bigwedge V$ endowed with this scalar product is called Grassmann algebra and will be denoted $\bigwedge(V, g)$.

On $\bigwedge V$ and $\bigwedge(V, g)$ there are two important involutive morphisms:

(i) Main automorphism $\widehat{A} : \bigwedge V \to \bigwedge V$,

$$\begin{gathered} (A \wedge B)\widehat{\ } = \widehat{A} \wedge \widehat{B}, \; A, B \in \bigwedge V \; ; \\ \widehat{A} = A \;\text{ if }\; A \in \bigwedge^0 V, \; \widehat{A} = -A \;\text{ if }\; A \in \bigwedge^1 V \; . \end{gathered} \tag{2.10}$$

[4] $\bigwedge V$ is what old physics textbooks call the algebra of antisymmetric tensors.

(ii) Reversion $\sim: \bigwedge V \to \bigwedge V$,

$$(A \wedge B)^{\sim} = \widetilde{B} \wedge \widetilde{A}, \; A, B \in \bigwedge V \; ; \qquad \widetilde{A} = A \;\text{ if }\; A \in \bigwedge^0 V \oplus \bigwedge^1 V \; . \tag{2.11}$$

We define also:

(iii) Conjugation: $-: \bigwedge V \to \bigwedge V$,

$$\overline{A} = (\widehat{A})^{\sim} = (\widetilde{A})^{\wedge}, \; \forall A \in \bigwedge V. \tag{2.12}$$

We introduce now the important concepts of left contraction $\rfloor : \bigwedge V \times \bigwedge V \to \bigwedge V$ and right contraction $\lfloor: \bigwedge V \times \bigwedge V \to \bigwedge V$ through the definitions

$$G(A \rfloor B, C) = G(B, \widetilde{A} \wedge C); \; G(A \lfloor B, C) = G(A, C \wedge \widetilde{B}), \; \forall C \in \bigwedge V \; . \tag{2.13}$$

$\rfloor$ and $\lfloor$ satisfy the rules

$$\begin{aligned} &1.\; x \rfloor y = x.y = g(x, y); \; x \lfloor y = x.y = g(x, y) \; ; \\ &2.\; x \rfloor (A \wedge B) = (x \rfloor A) \wedge B + \widehat{A} \wedge (x \rfloor B) \; ; \\ &3.\; (A \wedge B) \lfloor x = A \wedge (B \lfloor x) + (A \lfloor x) \wedge \widehat{B} \; ; \\ &4.\; (A \wedge B) \rfloor C = A \rfloor (B \rfloor C); \quad A \lfloor (B \wedge C) = (A \lfloor B) \lfloor C \; ; \end{aligned} \tag{2.14}$$

where $x, y \in \bigwedge^1 V, \; A, B, C \in \bigwedge V$.

The notation $A.B$ will be used for contractions when it is clear from the context which factor is the contractor and which factor is being contracted. When just one of the factors is homogeneous, it is understood to be the contractor. When both factors are homogeneous we agree that the one with the lowest degree is the contractor, so that for $A \in \bigwedge^r V, B \in \bigwedge^s V$ we have $A.B = A \rfloor B$ if $r \leq s$, $A.B = A \lfloor B$, if $r \geq s$. From the definitions and eq.(2.14) we easily verify that

$$s \rfloor A = 0, \; A \lfloor s = 0 \;\; \forall s \in \mathbb{R}, A \in \bigwedge V \; . \tag{2.15}$$

We are now ready to present the definition of the real *Clifford* algebra $\mathcal{C}\ell(V, g)$ associated with the pair (V, g). In order to do that we define the *Clifford product* (denoted by juxtaposition of symbols) between $x \in V$ and $A \in \bigwedge V$ by

$$xA = x \rfloor A + x \wedge A = x.A + x \wedge A$$

and extend this product by linearity and associativity to all of $\bigwedge V$.

Equipped with the Clifford product $\bigwedge V$ becomes isomorphic to the Clifford algebra $\mathcal{C}\ell(V, g)$.[5] Observe that $\bigwedge V$ equipped with the exterior product and $\mathcal{C}\ell(V, g)$ equipped with the Clifford product are, of course, not isomorphic as algebras. However, $\bigwedge V$ and $\mathcal{C}\ell(V, g)$ are isomorphic as linear spaces over $\mathbb{R}$.

Consider the basis of $\bigwedge V$ given by (2.5) and suppose that

$$g(E_i, E_j) = \begin{cases} +1, & i = j = 1, \ldots, p; \\ -1, & i = j = p + 1, \ldots, p + q; \\ 0, & \text{otherwise.} \end{cases} \tag{2.16}$$

[5] We can show that $\mathcal{C}\ell(V, g) = TV/J$ where J is the bilateral ideal on TV generated by elements of the form $a \otimes b + b \otimes a - 2g(a, b), a, b \in V \subset TV$ [1].

Then it is clear that $E_iE_j = E_i.E_j + E_i \wedge E_j = E_i \wedge E_j$ for all $i \neq j$. Since $\bigwedge V$ and $\mathcal{C}\ell(V,g)$ are isomorphic as linear spaces we can write for $X \in \mathcal{C}\ell(V,g)$

$$X = S + X_i E^i + \frac{1}{2} X_{ij} E^i E^j + \frac{1}{3!} X_{ijk} E^i E^j E^k + \cdots P E^1 E^2 E^3 \ldots E^n, \tag{2.17}$$

where $E^i.E_j = \delta^i_j$ and $\{E^i\}$, $i = 1, \ldots, n$ is called the reciprocal basis of V. Also $S, X_i, X_{ij}, \ldots, P \in \mathbb{R}$ and $X_{ij} = -X_{ji}$, etc.

For g of signature (p,q) as in eq.(2.16) $\mathcal{C}\ell(V,g)$ is denoted $\mathcal{C}\ell_{p,q}$. Using the projector operator $\langle\rangle_k$ defined above we can show that the contraction $A_r.B_s$, for $A_r \in \bigwedge^r V \subset \mathcal{C}\ell_{p,q}$, $B_s \in \bigwedge^s V \subset \mathcal{C}\ell_{p,q}$ is given by

$$A_r.B_s = \begin{cases} \langle A_r B_s \rangle_{|r-s|} & \text{if} \quad r, s > 0 \\ 0 \quad \text{if} \quad r = 0 \quad \text{or} \quad s = 0 \end{cases} . \tag{2.18}$$

Eq.(2.18) defines then an inner product in $\mathcal{C}\ell_{p,q}$.

We now define the Hodge star operator $\star : \mathcal{C}\ell(V,g) \to \mathcal{C}\ell(V,g)$ by

$$\star\, A = \widetilde{A}\, E_{n+1}, \quad A \in \mathcal{C}\ell(V,g). \tag{2.19}$$

A simple calculation shows that $\star|_{\bigwedge^p}$ maps $\bigwedge^p V \to \bigwedge^{n-p} V$ for $p = 0, 1, \ldots, n$. We observe that $\mathcal{C}\ell_{p,q}$ is a Z_2-graded algebra. This means the following. Let $\mathcal{C}\ell^+_{p,q}(\mathcal{C}\ell^-_{p,q})$ denote the set of even (odd) multivectors of $\mathcal{C}\ell_{p,q}$, *i.e.*, elements of $\bigwedge^{2r} V \subset \mathcal{C}\ell_{p,q}(\bigwedge^{2r+1} \subset \mathcal{C}\ell_{p,q})$. We have $\mathcal{C}\ell^+_{p,q}\mathcal{C}\ell^+_{p,q} \subset \mathcal{C}\ell^+_{p,q}$, $\mathcal{C}\ell^-_{p,q}\mathcal{C}\ell^-_{p,q} \subset \mathcal{C}\ell^+_{p,q}$, $\mathcal{C}\ell^+_{p,q}\mathcal{C}\ell^-_{p,q} \subset \mathcal{C}\ell^-_{p,q}$, $\mathcal{C}\ell^-_{p,q}\mathcal{C}\ell^+_{p,q} \subset \mathcal{C}\ell^-_{p,q}$. The $\mathcal{C}\ell^+_{p,q}$ is a sub-algebra of $\mathcal{C}\ell_{p,q}$, called the even sub-algebra of $\mathcal{C}\ell_{p,q}$. All Clifford algebras $\mathcal{C}\ell_{p,q}$ are isomorphic to matrix algebras over the fields $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, respectively the real, complex and quaternion fields. We can find in [9] tables giving the representations of $\mathcal{C}\ell_{p,q}$ as matrix algebras. For what follows we need to know the following representations:

$$\begin{array}{lcl} \text{Complex numbers} & - & \mathcal{C}\ell_{0,1} \simeq \mathbb{C} \\ \text{Quaternions} & - & \mathcal{C}\ell_{0,2} \simeq \mathbb{H} \\ \text{Pauli algebra} & - & \mathcal{C}\ell_{3,0} \simeq M_2(\mathbb{C}) \\ \text{Spacetime algebra} & - & \mathcal{C}\ell_{1,3} \simeq M_2(\mathbb{H}) \\ \text{Majorana algebra} & - & \mathcal{C}\ell_{3,1} \simeq M_4(\mathbb{R}) \\ \text{Dirac algebra} & - & \mathcal{C}\ell_{4,1} \simeq M_4(\mathbb{C}) \end{array} \tag{2.20}$$

Since it is a theorem that $\mathcal{C}\ell^+_{p,q} \simeq \mathcal{C}\ell_{q,p-1}$ for $p \geq 1$ and $\mathcal{C}\ell^+_{p,q} \simeq \mathcal{C}\ell_{p,q-1}$ for $q \geq 1$ we have the following useful *identifications* to be used later:

$$\mathcal{C}\ell^+_{1,3} \simeq \mathcal{C}\ell^+_{3,1} \simeq \mathcal{C}\ell_{3,0};\ \mathcal{C}\ell^+_{4,1} \simeq \mathcal{C}\ell_{1,3};\ \mathcal{C}\ell^+_{3,0} \simeq \mathbb{H};\ \mathbb{H}^+ \simeq \mathbb{C}. \tag{2.21}$$

A very important result is that the Dirac algebra is the tensor complexification of both $\mathcal{C}\ell_{1,3}$ and $\mathcal{C}\ell_{3,1}$, *i.e.*,

$$\mathcal{C}\ell_{4,1} \simeq \mathbb{C} \otimes \mathcal{C}\ell_{1,3};\quad \mathcal{C}\ell_{4,1} \simeq \mathbb{C} \otimes \mathcal{C}\ell_{3,1} \tag{2.22}$$

Since it is a well known result that $\mathbb{H}$ is represented by a subset of invertible two by two complex matrices belonging to $M_2(\mathbb{C})$, eq.(2.21) and eq.(2.22) show that $\mathcal{C}\ell_{1,3}$ has also a complex 4×4 matrix representation which can be made identical to the $M_4(\mathbb{C})$ representation of $\mathcal{C}\ell_{4,1}$.

Before ending this section we recall that the Clifford product between two general elements $A, B \in \mathcal{C}\ell_{p,q}$ can be written

$$\begin{aligned} AB &= \sum_{r,s} \langle A\rangle_r \langle B\rangle_s = \sum_{r,s} A_r B_s \\ &= \sum_{r,s} (\langle A_r B_s\rangle_{|r-s|} + \langle A_r B_s\rangle_{|r-s|+2} + \cdots + \langle A_r B_s\rangle_{r+s}). \end{aligned} \tag{2.23}$$

We define also the norm of a multivector $A \in \mathcal{C}\ell_{p,q}$ by

$$|A|^2 = \langle \widetilde{A} A\rangle_0. \tag{2.24}$$

If $A \in \mathcal{C}\ell_{3,0}$ is homogeneous and $|A|^2 \neq 0$, the inverse of A is[6]

$$A^{-1} = \widetilde{A}/|A|^2, \quad A^{-1}A = AA^{-1} = 1. \tag{2.25}$$

2.2. The spacetime and Pauli algebras

We call $I\!R^{1,3} = (I\!R^4, g)$ where $g : I\!R^4 \times I\!R^4 \to I\!R$ is a Lorentzian metric of signature $(1,3)$. $I\!R^{1,3}$ is called the Minkowski vector space. Let $\{E_\mu\}$, $\mu = 0,1,2,3$ be a basis of $I\!R^4$; we have

$$g(E_\mu, E_\nu) = \eta_{\mu\nu} = \begin{cases} +1 & \mu = \nu = 0; \\ -1 & \mu = \nu = 1,2,3; \\ \ 0 & \text{otherwise.} \end{cases} \tag{2.26}$$

The fundamental relation generating the **spacetime algebra** $\mathcal{C}\ell_{1,3}$ is then

$$E_\mu E_\nu + E_\nu E_\mu = 2\eta_{\mu\nu}. \tag{2.27}$$

Eq.(2.27) is identical to the relation satisfied by the famous Dirac (gamma) matrices and indeed we know from Section 2.1 that the E_μ have a complex matrix representation in $M_4(\mathbb{C})$. Naturally, $\dim \mathcal{C}\ell_{1,3} = 16$. The pseudoscalar of $\mathcal{C}\ell_{1,3}$ will be denoted by $E_5 = E_0E_1E_2E_3$ and $E_5^2 = -1$. E_5 anticommutes with odd multivectors and commutes with even multivectors. We call $\{E^\mu\}$, $\mu = 0,1,2,3$, such that $E^\mu . E_\nu = \delta^\mu_\nu$ the reciprocal basis to $\{E_\mu\}$.

We call the pair $I\!R^{1,3*} = (I\!R^4, g^{-1})$ the dual space of $(I\!R^4, g)$ and call $\{\Gamma_\mu\}$, $\mu = 0,1,2,3$ the dual basis to $\{E_\mu\}$. Analogously, $\{\Gamma^\mu\}$, $\mu = 0,1,2,3$ such that $g^{-1}(\Gamma^\mu, \Gamma_\nu) = \delta^\mu_\nu$ is the reciprocal basis to $\{\Gamma_\mu\}$. The Clifford algebra associated to $I\!R^{1,3*}$ will be denoted $*\mathcal{C}\ell_{1,3} \simeq \mathcal{C}\ell_{1,3} \simeq M_2(I\!H)$. Of course we have the fundamental relation

$$\Gamma^\mu\Gamma^\nu + \Gamma^\nu\Gamma^\mu = 2\eta^{\mu\nu}, \tag{2.28}$$

where $\eta^{\mu\nu} = \eta_{\mu\nu}$.

The **Pauli algebra** $\mathcal{C}\ell_{3,0}$ is the Clifford algebra of $(I\!R^3, g_E)$, *i.e.* of Euclidean space equipped with the Euclidean metric g_E. If $\{\vec{\Sigma}_i\}$, $i = 1,2,3$ is an orthonormal basis of $I\!R^3$, *i.e.* $\vec{\Sigma}_i . \vec{\Sigma}_j = g_E(\vec{\Sigma}_i, \vec{\Sigma}_j) = \delta_{ij}$ then the Clifford algebra $\mathcal{C}\ell_{3,0}$ is generated by the fundamental relation

$$\vec{\Sigma}_i\vec{\Sigma}_j + \vec{\Sigma}_j\vec{\Sigma}_i = 2\delta_{ij}. \tag{2.29}$$

[6] The calculation of A^{-1} (when it exists) for a general $A \in \mathcal{C}\ell_{p,q}$ is not so simple.

$I = \vec{\Sigma}_1\vec{\Sigma}_2\vec{\Sigma}_3$ is the pseudoscalar of $\mathcal{C}\ell_{3,0}$. We verify that $I^2 = -1$ and that I commutes with all $X \in \mathcal{C}\ell_{3,0}$, so that I is like $\mathrm{i} = \sqrt{-1}$. A basis for $\mathcal{C}\ell_{3,0}$ is $(1, \vec{\Sigma}_i, \vec{\Sigma}_i\vec{\Sigma}_j,\ \vec{\Sigma}_1\vec{\Sigma}_2\vec{\Sigma}_3)$. Taking into account that

$$\star\,(\vec{\Sigma}_1\vec{\Sigma}_2) = \vec{\Sigma}_2\vec{\Sigma}_1 I = \vec{\Sigma}_3;\ \star(\vec{\Sigma}_1\vec{\Sigma}_3) = \vec{\Sigma}_3\vec{\Sigma}_1 I = -\vec{\Sigma}_2;\ \star(\vec{\Sigma}_2\vec{\Sigma}_3) = \vec{\Sigma}_3\vec{\Sigma}_2 I = \vec{\Sigma}_1; \tag{2.30}$$

we see that $\forall\ X \in \mathcal{C}\ell_{3,0}$ can be written as

$$X = (s + Ip) + (\vec{A} + I\vec{B}) \quad s, p \in \mathbb{R};\ \vec{A}, \vec{B} \in \mathbb{R}^3; \tag{2.31}$$

i.e., X is "formally the sum" of a "complex number" and a "complex vector". From eq.(2.21) we see that $\mathcal{C}\ell_{3,0} \simeq \mathcal{C}\ell^+_{1,3}$. We can exhibit this isomorphism by identifying $\Sigma_k = E_k E_0 \simeq \vec{\Sigma}_k$, $k = 1,2,3$ and where $E_0^2 = 1$ is timelike. We define in $\mathcal{C}\ell_{3,0}$ the operator of spatial inversion. For $X \in \mathcal{C}\ell_{3,0}$ as in eq.(2.31),

$$* : X \mapsto X^* = (s + Ip) - (\vec{A} + I\vec{B}). \tag{2.32}$$

With the above identification $p = p^\mu E_\mu \in \bigwedge^1(R^{1,3}) \subset \mathcal{C}\ell_{1,3}$ can be represented in $\mathcal{C}\ell^+_{1,3} \simeq \mathcal{C}\ell_{3,0}$ by

$$p \mapsto pE_0 = p^0 + p^i\Sigma_i = p^0 + \vec{p}. \tag{2.33}$$

For $f = \frac{1}{2} f^{\mu\nu} E_\mu E_\nu \in \bigwedge^2(\mathbb{R}^{1,3}) \subset \mathcal{C}\ell_{1,3}$ where

$$f^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta} f_{\alpha\beta} = \begin{pmatrix} 0 & -\mathcal{E}_1 & -\mathcal{E}_2 & -\mathcal{E}_3 \\ \mathcal{E}_1 & 0 & -\mathcal{B}_3 & \mathcal{B}_2 \\ \mathcal{E}_2 & \mathcal{B}_3 & 0 & -\mathcal{B}_1 \\ \mathcal{E}_3 & -\mathcal{B}_2 & \mathcal{B}_1 & 0 \end{pmatrix} \tag{2.34}$$

we can write

$$f = -f^{0i} E_i E_0 + f^{ij} E_i E_j = -f^{0i}\Sigma_i + f^{ij}\Sigma_i\Sigma_j$$

and taking into account (2.29) and (2.30) we can write

$$f = \vec{\mathcal{E}} + I\vec{\mathcal{B}} \qquad I = \Sigma_1\Sigma_2\Sigma_3. \tag{2.35}$$

Eq.(2.35) shows that any bivector $\vec{F} \in \bigwedge^2(\mathbb{R}^{1,3}) \subset \mathcal{C}\ell_{1,3}$ can be represented in $\mathcal{C}\ell_{3,0} \simeq \mathcal{C}\ell^+_{1,3}$ by a sum of a vector and a Pauli bivector or a "complex vector".

We define next the Spin groups $\mathrm{Spin}_+(3,0) \simeq SU(2)$ and $\mathrm{Spin}_+(1,3) \simeq SL(2,\mathbb{C})$, which are respectively the covering groups of $SO_+(3)$, the special rotation group and $SO_+(1,3) \simeq \mathcal{L}^\uparrow_+$, the restricted orthochronous Lorentz group. We have

$$\mathrm{Spin}_+(3,0) = \{R \in \mathcal{C}\ell^+_{3,0} \mid \quad |R| = 1\}, \tag{2.36}$$

$$\mathrm{Spin}_+(1,3) = \{U \in \mathcal{C}\ell^+_{1,3} | \quad |U| = 1\}. \tag{2.37}$$

An arbitrary Lorentz rotation is given for $a \in \mathcal{C}\ell_{1,3}$. by $a \mapsto Ua\widetilde{U} = UaU^{-1}$, $U \in \mathrm{Spin}_+(1,3)$ We can prove that any $U \in \mathrm{Spin}_+(1,3)$ can be written in the form $U = \pm\mathrm{e}^f, f \in \bigwedge^2(R^{1,3})$, and the choice of the sign can always be positive except when $U = -\mathrm{e}^f$ with $f^2 = 0$. When $f^2 > 0$, U is a boost and when $f^2 < 0$, U is a spatial rotation. We end this Section with the definitions of minimal left *ideals* of $\mathcal{C}\ell_{p,q}$ and of *geometrically equivalent* ideals.

We say that $e \in \mathcal{C}\ell_{p,q}$ is *idempotent* if $\mathrm{e}^2 = e$; it is called a *primitive* idempotent if it *cannot* be written as a sum of two mutually annihilating idempotents, *i.e.*, $e \neq e' + e''$, with $(e')^2 = e'$, $(e'')^2 = e''$, $e'e'' = e''e' = 0$.

The sub-algebra $I_e \subset \mathcal{C}\ell_{p,q}$ is called a *left ideal* of $\mathcal{C}\ell_{p,q}$ if $\forall \psi \in I_e$ and $\forall X \in \mathcal{C}\ell_{p,q}$ we have $X\psi \in I_e$ (a similar definition exists for right ideals).

An ideal is said to be *minimal* if it contains only *trivial* sub-ideals. It can be shown that the minimal left ideals of $\mathcal{C}\ell_{p,q}$ are of the form $\mathcal{C}\ell_{p,q}e$, where e is a primitive idempotent.

Consider now $\mathcal{C}\ell_{1,3}$ and the orthonormal bases $\Sigma = \{E_\mu\}$ and $\dot{\Sigma} = \{\dot{E}_\mu\}$ where $\dot{E}_\mu = UE_\mu\widetilde{U}$, $U \in \mathrm{Spin}_+(1,3)$. We can easily verify that the following elements are primitive idempotents of $\mathcal{C}\ell_{1,3}$:

$$\begin{array}{lll} e_\Sigma = \frac{1}{2}(1 + E_0); & e'_\Sigma = \frac{1}{2}(1 + E_3E_0); & e''_\Sigma = \frac{1}{2}(1 + E_1E_2E_3); \\ e_{\dot{\Sigma}} = \frac{1}{2}(1 + \dot{E}_0); & e'_{\dot{\Sigma}} = \frac{1}{2}(1 + \dot{E}_3\dot{E}_0); & e''_{\dot{\Sigma}} = \frac{1}{2}(1 + \dot{E}_1\dot{E}_2\dot{E}_3). \end{array} \tag{2.38}$$

It is trivial to verify that e_Σ and $e_{\dot{\Sigma}}$ are related by

$$e_{\dot{\Sigma}} = Ue_\Sigma U^{-1}, \qquad u \in \mathrm{Spin}_+(1,3). \tag{2.39}$$

There is no element $U \in \mathrm{Spin}_+(1,3)$ relating, *e.g.* e'_Σ with e_Σ. Consider now the ideals $I_\Sigma = \mathcal{C}\ell_{1,3}e_\Sigma$ and $I_{\dot{\Sigma}} = \mathcal{C}\ell_{1,3}e_{\dot{\Sigma}}$. We say that I_Σ and $I_{\dot{\Sigma}}$ are *geometrically equivalent* if e_Σ and $e_{\dot{\Sigma}}$ are related by eq.(2.39). Since $I_\Sigma = \mathcal{C}\ell_{1,3}e_\Sigma$ and $I_{\dot{\Sigma}} = \mathcal{C}\ell_{1,3}e_{\dot{\Sigma}}$ and since $\mathcal{C}\ell_{1,3}U \simeq \mathcal{C}\ell_{1,3}$ $\forall U \in \mathrm{Spin}_+(1,3)$, we can write

$$I_{\dot{\Sigma}} = I_\Sigma U^{-1} \tag{2.40}$$

Eq.(2.40) defines a correspondence between elements of ideals that are geometrically equivalent. The quotient set $\{I_\Sigma\}/\mathcal{R}$ where $\mathcal{R}$ is the equivalence relation given by eq.(2.40) is called the space of the Dirac algebraic spinors Σ. Of course in the basis $\dot{\Sigma}$ the spinor is represented by $\psi_{\dot{\Sigma}}$ and

$$\psi_{\dot{\Sigma}} = \psi_\Sigma U^{-1} \tag{2.41}$$

Section 2.4, where we introduce the *fundamental* concept of Dirac-Hestenes spinors, will clarify the meaning of the above definitions.

2.3. Dirac algebra $\mathcal{C}\ell_{4,1}$, its relation with $\mathcal{C}\ell_{1,3}$ and Dirac-Hestenes spinors

Consider the vector space $I\!R^{4,1} = (I\!R^5, g)$ and let $\{E_a\}$, $a = 0, 1, \dots, 4$ be an orthonormal basis:

$$g(E_a, E_b) = \begin{cases} 1, & a = b = 1,2,3,4; \\ -1, & a = b = 0; \\ 0, & \text{otherwise.} \end{cases} \tag{2.42}$$

Let $\mathcal{C}\ell_{4,1}$ be the Clifford algebra of $I\!R^{4,1}$ and $I = E_0E_1E_2E_3E_4$ the corresponding pseudoscalar. Note that $I^2 = -1$ and that $E_aI = IE_a$, and thus $I\!R \oplus I I\!R$ is the center of $\mathcal{C}\ell_{4,1}$ and the pseudoscalar I plays therefore the role of the imaginary unit (as I in the case of the Pauli algebra).

Let us define

$$\Gamma_\mu = E_\mu E_4, \quad \mu = 0,1,2,3. \tag{2.43}$$

Then,

$$\Gamma_\mu\Gamma_\nu + \Gamma_\nu\Gamma_\mu = 2\eta_{\mu\nu}. \tag{2.44}$$

One can easily see from (2.43), (2.44) and with I playing the role of the imaginary unity ($\mathrm{i} = \sqrt{-1}$) that $\mathcal{C}\ell_{4,1}$ is isomorphic to the complexified spacetime algebra, *i.e.*,

$$\mathcal{C}\ell_{4,1} \simeq \mathbb{C} \otimes \mathcal{C}\ell_{1,3}. \tag{2.45}$$

Indeed, each $X \in \mathcal{C}\ell_{4,1}$ can be written

$$\begin{aligned} X &= a + X^a E_a + \frac{1}{2} X^{ab} E_a E_b + \frac{1}{3!} X^{abc} E_a E_b E_c + \frac{1}{4!} X^{abcd} E_a E_b E_c E_d + Ib \\ &= (a + Ib) + X^\mu E_\mu - (Y^\mu E_\mu I) - I X^S E_0 E_1 E_2 E_3 + \cdots \\ &= (a + Ib) + (A^\mu + IB^\mu)\Gamma_\mu + \frac{1}{2!}(A^{\mu\nu} + IB^{\mu\nu})\Gamma_\mu \Gamma_\nu + \cdots + p\Gamma_0\Gamma_1\Gamma_2\Gamma_3 \end{aligned} \tag{2.46}$$

$a, b, X^\mu, Y^\mu, \ldots, B^{\mu\nu}, \ldots \in \mathbb{R}$.

Moreover, the even sub-algebra of $\mathcal{C}\ell_{4,1}$ is isomorphic to $\mathcal{C}\ell_{1,3}$, *i.e.*,

$$\mathcal{C}\ell_{4,1}^{+} \simeq \mathcal{C}\ell_{1,3}.$$

$\mathcal{C}\ell_{4,1}$, the complexified spacetime algebra is the well known Dirac algebra studied in physics textbooks. Indeed $\mathcal{C}\ell_{4,1}$ is isomorphic to $M_4(\mathbb{C})$, the algebra of 4×4 matrices over the complex. One representation (the standard one) of the Γ_μ defined by eq.(2.43) is

$$\Gamma_0 \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}; \quad \Gamma_1 \leftrightarrow \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix};$$
$$\Gamma_2 \leftrightarrow \begin{pmatrix} 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & -\mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \end{pmatrix}; \quad \Gamma_3 \leftrightarrow \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{2.47}$$

Consider the idempotent $f = \frac{1}{2}(1 + \Gamma_0)\frac{1}{2}(1 + I\Gamma_1\Gamma_2) = e_\Sigma \frac{1}{2}(1 + I\Gamma_1\Gamma_2)$ where $e_\Sigma = \frac{1}{2}(1 + \Gamma_0)$ is a primitive idempotent of $\mathcal{C}\ell_{4,1}^{+} \simeq \mathcal{C}\ell_{1,3}$. It generates the left minimal ideal $I_{4,1}^{\Sigma} = \mathcal{C}\ell_{4,1} f$ and we can easily verify by explicit computation that

$$I_{4,1}^{\Sigma} = \mathcal{C}\ell_{4,1} f \simeq \mathcal{C}\ell_{4,1}^{+} f. \tag{2.48}$$

Consider now the usual Dirac spinor $|\Psi\rangle \in \mathbb{C}^4$. There is an obvious isomorphism between $\mathbb{C}^4$ and minimal left ideals of $M_4(\mathbb{C})$, given by

$$\mathbb{C}^4 \ni |\Psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \leftrightarrow \begin{pmatrix} \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \\ \psi_4 & 0 & 0 & 0 \end{pmatrix} = \Psi \in \text{ minimal left ideal of } M_4(\mathbb{C}) \tag{2.49}$$

One can, of course, work with Ψ instead of $|\Psi\rangle$ and since $M_4(\mathbb{C})$ is a representation of the Dirac algebra $\mathcal{C}\ell_{4,1} \simeq \mathbb{C} \otimes \mathcal{C}\ell_{1,3}$ we can work with the corresponding ideal $I^{\Sigma}_{4,1}$ of the Dirac algebra. The isomorphisms discussed above tell us that

$$I^{\Sigma}_{4,1} = \mathcal{C}\ell_{4,1}f = (\mathbb{C} \otimes \mathcal{C}\ell_{1,3})f \simeq \mathcal{C}\ell^{+}_{4,1}f \simeq \mathcal{C}\ell_{1,3}f = (\mathcal{C}\ell_{1,3}e_{\Sigma})\frac{1}{2}(1 + I\Gamma_1\Gamma_2). \tag{2.50}$$

Note that in the last equality we have a minimal left ideal $\mathcal{C}\ell_{1,3}e_{\Sigma}$ of the spacetime algebra. Moreover we have

$$If = \Gamma_2\Gamma_1 f. \tag{2.51}$$

These results mean that we can work with the ideal $\mathcal{C}\ell_{1,3}e_{\Sigma}$ once we identify $\Gamma_2\Gamma_1$ as playing in $\mathcal{C}\ell_{1,3}$ the role of the imaginary unit. We can verify by explicit calculation that

$$\mathcal{C}\ell_{1,3}e_{\Sigma} = \mathcal{C}\ell^{+}_{1,3}e_{\Sigma}. \tag{2.52}$$

We see that what the idempotent makes is to "kill" redundant degrees of freedom. Since $\dim \mathcal{C}\ell^{+}_{1,3} = \dim \mathcal{C}\ell_{3,0} = 8$ we can work with $\mathcal{C}\ell^{+}_{1,3}$ instead of $\mathcal{C}\ell_{1,3}e_{\Sigma}$ (this is not the case for $\mathcal{C}\ell_{1,3}$ or $\mathbb{C} \otimes \mathcal{C}\ell_{1,3}$ since $\dim \mathcal{C}\ell_{1,3} = 16$ and $\dim(\mathbb{C} \otimes \mathcal{C}\ell_{1,3}) = 32$). We have thus established the isomorphism

$$\mathbb{C}^4 \simeq \mathcal{C}\ell^{+}_{1,3}. \tag{2.53}$$

We shall call Ψ_{Σ} the representative of $|\Psi\rangle$ in $\mathcal{C}\ell_{4,1}$. It is related to $\psi_{\Sigma} \in (\mathcal{C}\ell^{+}_{4,1})^{+} \simeq \mathcal{C}\ell^{+}_{1,3}$ by

$$\Psi_{\Sigma} = \psi_{\Sigma}\frac{1}{2}(1 + \Gamma_0)\frac{1}{2}(1 + I\Gamma_1\Gamma_2). \tag{2.54}$$

Such a ψ_{Σ} will be called the "representative" of a Dirac-Hestenes spinor in the basis $(\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3)$ of $\mathcal{C}\ell^{+}_{4,1} \simeq \mathcal{C}\ell_{1,3}$. Its standard matrix representation is

$$\psi_{\Sigma} \;\leftrightarrow\; \begin{pmatrix} \psi_1 & -\psi_2^* & \psi_3 & \psi_4^* \\ \psi_2 & \psi_1^* & \psi_4 & \psi_3^* \\ \psi_3 & \psi_4^* & \psi_1 & -\psi_2^* \\ \psi_4 & -\psi_3^* & \psi_2 & \psi_1^* \end{pmatrix}. \tag{2.55}$$

A *Dirac-Hestenes* spinor is an element of the quotient set $\mathcal{C}\ell_{1,3}/\mathcal{R}$ such that given two orthonormal basis $\Sigma = \{\Gamma_\mu\}$, $\dot{\Sigma} = \{\Gamma'_\mu\}$ of $\mathbb{R}^{1,3} \subset \mathcal{C}\ell_{1,3}$, $\psi_{\Sigma} \in \mathcal{C}\ell^{+}_{1,3}$, then $\psi_{\Sigma} \sim \psi_{\dot{\Sigma}}(\mathrm{mod}\mathcal{R})$ if and only if $\psi_{\dot{\Sigma}} = \psi_{\Sigma}U^{-1}$ with $\Sigma = \mathcal{L}(\Sigma) = U\Sigma U^{-1}$, $U \in \mathrm{Spin}_{+}(1,3)$, $\mathcal{L} \in SO_{+}(1,3)$ and $\mathcal{H}(U) = \mathcal{L}$ where $\mathcal{H} : \mathrm{Spin}_{+}(1,3) \to SO_{+}(1,3)$ is the universal double covering of $SO_{+}(1,3)$. We already said that ψ_{Σ} is the representative of the Dirac-Hestenes spinors in the basis Σ. When no confusion arises we shall write only ψ instead of ψ_{Σ}. $\psi_{\Sigma}\frac{1}{2}(1 + \Gamma_0)$ is the Dirac algebraic spinor introduced in Section 2.2. From now on we work with $\mathcal{C}\ell_{1,3}$. Then $\psi \in \mathcal{C}\ell^{+}_{1,3}$ can be written as

$$\psi = S + F + \Gamma_5 P,\; \Gamma_5 = \Gamma_0\Gamma_1\Gamma_2\Gamma_3, \tag{2.56}$$

$S, P \in \mathbb{R}$, and $F \in \bigwedge^2(R^{1,3}) \subset \mathcal{C}\ell_{1,3}$ is a bivector.

Suppose now that ψ is nonsingular, *i.e.*, $\psi\widetilde{\psi} \neq 0$. Since $\psi \in \mathcal{C}\ell^{+}_{1,3}$ we have

$$\psi\widetilde{\psi} = \sigma + \Gamma_5\omega,\; \sigma, \omega \in \mathbb{R}. \tag{2.57}$$

Define $\rho = \sqrt{\sigma^2 + w^2}$, $\tan\beta = \omega/\sigma$. Then we have

$$\psi = \sqrt{\rho}\mathrm{e}^{\Gamma_5\beta/2}R, \tag{2.58}$$

where $R \in \mathrm{Spin}_+(1,3)$, $\rho \in I\!R^+$ and $0 \leq \beta < 2\pi$ is the so called Ivon-Takabayasi angle. This is the canonical decomposition of Dirac-Hestenes spinors and reveals the secret geometrical meaning of spinors, for if $X \in I\!R^{1,3} \subset \mathcal{C}\ell_{1,3}$

$$\psi X \widetilde{\psi} = \rho R X \widetilde{R} = \rho Y \quad , \quad Y = R X \widetilde{R} \in I\!R^{1,3} \subset \mathcal{C}\ell_{1,3}, \tag{2.59}$$

i.e., a Dirac Hestenes spinor acting on a vector produces a Lorentz rotation plus a dilation of the vector.

A Weyl spinor $\psi \in \mathcal{C}\ell_{1,3}/\mathcal{R}$ is such that its representative in a given frame Σ satisfy the condition [10]

$$\gamma_5 \psi = \pm \psi \gamma_{21}\,. \tag{2.60}$$

Such spinors are called positive and negative eigenstates of γ_5 and are denoted by ψ_+ (ψ_-). For a general $\psi \in \mathcal{C}\ell^+_{1,3}$ we can write

$$\psi_\pm = \frac{1}{2}[\psi \mp \gamma_5 \psi \gamma_{21}]. \tag{2.61}$$

2.4. The Clifford bundle of differential forms and the Spin-Clifford bundle

Let $\mathcal{M} = (M, g, D)$ be Minkowski spacetime, where (M, g) is a four dimensional time oriented and spacetime oriented Lorentzian manifold, with $M \simeq I\!R^4$ and with $g \in \sec(T^*M \times T^*M)$ being a Lorentzian metric of signature $(1,3)$.[7] $T^*M[TM]$ is the cotangent [tangent] bundle. $T^*M = \cup_{x\in M} T^*_x M$ $[TM = \cup_{x \in M} T_x M]$ and $T_xM \simeq T^*_x M \simeq I\!R^{1,3}$, the Minkowski vector space already defined above. D is the Levi-Civita connection of g, *i.e.*, $D(g) = 0$, $T(D) = 0$. Also $I\!R(D) = 0$, T and $I\!R$ being respectively the torsion and curvature tensors. Now, the *Clifford bundle* of differential forms $\mathcal{C}\ell(M)$ is the vector bundle of algebras $\mathcal{C}\ell(M) = \cup_{x\in M} \mathcal{C}\ell(T^*_x M)$ where $\forall x \in M$, $\mathcal{C}\ell(T^*_xM) \simeq \mathcal{C}\ell_{1,3}$, the spacetime algebra. As a linear space $\mathcal{C}\ell(T^*_xM)$ is isomorphic to the exterior algebra $\bigwedge(T^*_xM) \simeq \bigwedge(I\!R^{1,3*})$ of the space $I\!R^{1,3*}$ dual of $I\!R^{1,3}$. Then the so called Cartan bundle $\bigwedge(M) = \cup_{x\in M} \bigwedge(T^*_xM)$ can be thought as "embedded" in $\mathcal{C}\ell(M)$. In this way sections of $\mathcal{C}\ell(M)$ can be represented as a sum of inhomogeneous differential forms [2, 3].

Let $\{e_\mu\}$, $\mu = 0,1,2,3$, $e_\mu \in \sec TM$ be an orthonormal basis of TM, *i.e.*, $g(e_\mu, e_\nu) = \eta_{\mu\nu} = diag(1,-1,-1,-1)$ and let its dual basis be $\{\gamma^\mu\}$, $\mu = 0,1,2,3$, $\gamma^\mu \in \sec \bigwedge^1(M) \subset \sec \mathcal{C}\ell(M)$. Then if $g^{-1} \in \sec(TM \times TM)$ is the metric on T^*M, we have $g^{-1}(\gamma^\mu, \gamma^\nu) = \eta^{\mu\nu} = diag(1,-1,-1,-1)$. The fundamental Clifford product is generated by

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}. \tag{2.62}$$

We introduce also the reciprocal basis $\{\gamma_\mu\}$, $\mu = 0,1,2,3$, $\gamma_\mu . \gamma^\nu = \delta^\nu_\mu$, $\gamma_\mu \in \sec \bigwedge^1(M) \subset \sec \mathcal{C}\ell(M)$. Then $\mathcal{C} \in \sec \mathcal{C}\ell(M)$ can be written as

$$\mathcal{C} = s + v^\mu \gamma_\mu + \frac{1}{2} b^{\mu\nu} \gamma_\mu \gamma_\nu + \frac{1}{3!} a^{\mu\nu\rho} \gamma_\mu \gamma_\nu \gamma_\rho + p\gamma_5, \tag{2.63}$$

where $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$ and v, v^μ, $b^{\mu\nu}$, $a^{\mu\nu\rho}$, $\rho \in \sec \bigwedge^0(M) \subset \sec \mathcal{C}\ell(M)$.

Besides $\mathcal{C}\ell(M)$ we need to introduce another vector bundle, $\mathcal{C}\ell_{\mathrm{Spin}_+(1,3)}(M)$, called the Spin-Clifford bundle [2, 3], which is a quotient bundle,*i.e.*, $\mathcal{C}\ell_{\mathrm{Spin}_+(1,3)} = \mathcal{C}\ell(M)/\mathcal{R}$. This means that the sections of $\mathcal{C}\ell_{\mathrm{Spin}_+(1,3)}(M)$ are some special equivalence classes of

[7] Here sec means Section of a given bundle.

sections of the Clifford bundle, *i.e.*, they are equivalence sections of non-homogeneous differential forms. A given Section $\psi \in \sec \mathcal{C}\ell_{\mathrm{Spin}_+(1,3)}$ is then represented by ψ_Σ, $\psi_{\dot\Sigma}, \ldots \in \sec \mathcal{C}\ell(M)$ where $\Sigma, \dot\Sigma, \ldots$ are orthonormal bases of $\bigwedge^1(M) \subset \mathcal{C}\ell(M)$, with

$$\psi_{\dot\Sigma} = \psi_\Sigma R \tag{2.64}$$

and $\forall x \in M$, $R(x) \in \mathrm{Spin}_+(1,3)$.

Dirac-Hestenes spinor fields (DHSF) are sections of $\mathcal{C}\ell^+_{\mathrm{Spin}_+(1,3)}(M)$, the even sub-bundle of $\mathcal{C}\ell_{\mathrm{Spin}_+(1,3)}(M)$. The representative of a DHSF on $\mathcal{C}\ell(M)$ in the basis Σ is then

$$\psi_\Sigma = s + \frac{1}{2} b^{\mu\nu}\gamma_\mu\gamma_\nu + p\gamma_5 \tag{2.65}$$

The Hodge star map $\star : \bigwedge^p(M) \to \bigwedge^{4-p}(M)$ can be represented in $\mathcal{C}\ell(M)$ by the following algebraic operation:

$$\star A = \widetilde{A}\gamma_5, \tag{2.66}$$

for $A \in \sec \bigwedge^p(M) \subset \mathcal{C}\ell(M)$.

Let d and δ be respectively the *differential* and *Hodge codifferential* operators acting on sections of $\bigwedge(M) \subset \mathcal{C}\ell(M)$. We have

$$\mathrm{d} : \bigwedge^p(M) \to \bigwedge^{p+1}(M).$$

If $\sec \bigwedge^p(M) \ni \omega_p = \frac{1}{p!}\omega_{\alpha\beta}\ldots\gamma^\alpha \wedge \gamma^\beta \wedge \ldots$ then,

$$\mathrm{d}\omega_p = \frac{1}{p!} e_\mu(\omega_{\alpha\beta}\ldots)\gamma^\mu \wedge \gamma^\alpha \wedge \gamma^\beta \wedge \ldots \tag{2.67}$$

and $\mathrm{d}^2 = 0$. Also,

$$\delta : \bigwedge^p(M) \to \bigwedge^{p-1}(M), \quad \delta\omega_p = (-)^p \star^{-1} \mathrm{d} \star \omega_p, \tag{2.68}$$

where $\star\star^{-1} = \star^{-1}\star =$ identity and $\delta^2 = 0$.

The Dirac operator acting on sections of $\mathcal{C}\ell(M)$ is the invariant first order differential operator $\partial : \mathcal{C}\ell(M) \to \mathcal{C}\ell(M)$

$$\partial = \gamma^\mu D_{e_\mu} \tag{2.69}$$

and it holds the very important result (see *e.g.* [1])

$$\partial = \partial\wedge + \partial. = \mathrm{d} - \delta \tag{2.70}$$

For $\omega_p \in \sec \bigwedge^p(M) \subset \sec \mathcal{C}\ell(M)$

$$\begin{aligned} \partial\omega_p = \gamma^\mu D_{e_\mu}\omega_p &= \gamma^\mu \wedge (D_{e_\mu}\omega_p) + \gamma^\mu.(D_{e_\mu}\omega_p) \\ &= \partial \wedge \omega_p + \partial.\omega_p \end{aligned} \tag{2.71}$$

The operator $\Box = \partial\partial = (\mathrm{d} - \delta)(\mathrm{d} - \delta) = -(\mathrm{d}\delta + \delta\mathrm{d})$ is called Hodge Laplacian.

2.5. Maxwell theory in $\mathcal{C}\ell(M)$ and the Hertz potential

We shall need the concepts of inertial reference frames (I), observers and naturally adapted coordinate systems.

Let $\mathcal{M} = (M, g, D)$ be Minkowski spacetime. An *inertial reference frame* (irf) I is a timelike vector field $I \in \sec TM$ pointing into the future such that $g(I, I) = 1$ and $DI = 0$. Each integral line of I is called an inertial *observer*. The coordinate functions $\langle x^\mu \rangle$, $\mu = 0, 1, 2, 3$ of a chart of the maximal atlas of M are said to be a naturally adapted coordinate system to I (nacs/I) if $I = \partial/\partial x^0$ [11, 12]. Putting $I = e_0$ we can find $e_i = \partial/\partial x^i, i = 1, 2, 3$ such that $g(e_\mu, e_\nu) = \eta_{\mu\nu}$ and the coordinate functions x^μ are the usual Einstein-Lorentz ones and have a precise operational meaning: $x^0 = ct$,[8] where t is measured by "ideal clocks" at rest on I and synchronized "à la Einstein", $x^i, i = 1, 2, 3$ are determined with ideal rules [13]. (We use units where $c = 1$.)

Let $e_\mu \in \sec TM$ be an orthonormal basis $g(e_\mu, e_\nu) = \eta_{\mu\nu}$ and $e_\mu = \partial/\partial x^\mu (\mu, \nu = 0, 1, 2, 3)$. e_0 determines an IRF. Let $\gamma^\mu \in \sec \bigwedge^2(M) \subset \sec \mathcal{C}\ell(M)$ be the dual basis and let $\gamma_\mu = \eta_{\mu\nu}\gamma^\nu$ be the reciprocal basis to γ^μ, *i.e.*, $\gamma^\mu . \gamma_\nu = \delta^\mu_\nu$. We have $\gamma^\mu = \mathrm{d}x^\mu$.

As is well known the electromagnetic field is represented by a two-form $F \in \sec \bigwedge^2(M) \subset \sec \mathcal{C}\ell(M)$. We have

$$F = \frac{1}{2} F^{\mu\nu} \gamma_\mu \gamma_\nu, \; F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}, \tag{2.72}$$

where (E^1, E^2, E^3) and (B^1, B^2, B^3) are respectively the Cartesian components of the electric and magnetic fields.

Let $J \in \sec \bigwedge^1(M) \subset \sec \mathcal{C}\ell(M)$ be such that

$$J = J^\mu \gamma_\mu = \rho\gamma_0 + J^1\gamma_1 + J^2\gamma_2 + J^3\gamma_3, \tag{2.73}$$

where ρ and (J^1, J^2, J^3) are the Cartesian components of the charge and (3-dimensional) current densities. Recalling the definition of the operators d (eq.(2.67)) and δ (eq.(2.68)) we see that we can write Maxwell equations as

$$\mathrm{d}F = 0, \; \delta F = -J. \tag{2.74}$$

Since $\mathrm{d}F$ and δF are sections of $\mathcal{C}\ell(M)$ we can add the two equations in eq.(2.74) and get

$$(\mathrm{d} - \delta)F = J.$$

But from eq.(2.69), $\mathrm{d} - \delta = \partial$, the Dirac operator acting on sections of $\mathcal{C}\ell(M)$, and we get

$$\partial F = J \tag{2.75}$$

which may now be called *Maxwell equation*, instead of Maxwell equations.

We now write Maxwell equation in $\mathcal{C}\ell^+(M)$, the even sub-algebra of $\mathcal{C}\ell(M)$. The typical fiber of $\mathcal{C}\ell^+(M)$, which is a vector bundle, is isomorphic to the Pauli algebra (see Section 2.4).

[8]Here c is the constant called velocity of light in vacuum. In view of the superluminal and subluminal solutions of Maxwell equations found in this paper we don't think the terminology to be still satisfactory.

We put

$$\vec{\sigma}_i = \gamma_i\gamma_0,\ \mathbf{i} = \vec{\sigma}_1\vec{\sigma}_2\vec{\sigma}_3 = \gamma_0\gamma_1\gamma_2\gamma_3 = \gamma_5. \tag{2.76}$$

Recall that $\mathbf{i}$ commutes with bivectors and since $\mathbf{i}^2 = -1$ it acts like the imaginary unit $\mathrm{i} = \sqrt{-1}$ in $\mathcal{C}\ell^+(M)$. From eq.(2.69), using eq.(2.35) we get

$$F = \vec{E} + \mathbf{i}\vec{B} \tag{2.77}$$

with $\vec{E} = E^i\vec{\sigma}_i$, $\vec{B} = B^j\vec{\sigma}_j$, $i, j = 1, 2, 3$.

Now, since $\partial = \gamma_\mu\partial^\mu$ we get $\partial\gamma_0 = \partial/\partial x^0 + \vec{\sigma}_i\partial^i = \partial/\partial x^0 - \nabla$. Multiplying eq.(2.72) on the right by γ_0 we have

$$\partial\gamma_0\gamma_0F\gamma_0 = J\gamma_0,$$

$$(\partial/\partial x^0 - \nabla)(-\vec{E} + \mathbf{i}\vec{B}) = \rho + \vec{J}, \tag{2.78}$$

where we used $\gamma^0F\gamma_0 = -\vec{E} + \mathbf{i}\vec{B}$ and $\vec{J} = J^i\vec{\sigma}_i$.

From eq.(2.78) we have

$$-\partial_0\vec{E} + \mathbf{i}\partial_0\vec{B} + \nabla\vec{E} - \mathbf{i}\nabla\vec{B} = \rho + \vec{J}$$
$$-\partial_0\vec{E} + \mathbf{i}\partial_0\vec{B} + \nabla.\vec{E} + \nabla\wedge\vec{E} - \mathbf{i}\nabla.\vec{B} - \mathbf{i}\nabla\wedge\vec{B} = \rho + \vec{J}$$

Now we have

$$-\mathbf{i}\nabla\wedge\vec{A} \equiv \nabla\times\vec{A} \tag{2.79}$$

since the usual vector product between two vectors $\vec{a} = a^i\vec{\sigma}_i$, $\vec{b} = b^i\vec{\sigma}_i$ can be identified with the dual of the bivector $\vec{a}\wedge\vec{b}$ through the formula $\vec{a}\times\vec{b} = -\mathbf{i}(\vec{a}\wedge\vec{b})$. Observe that in this formalism $\vec{a}\times\vec{b}$ is a true vector and not the nonsense pseudo vector of the Gibbs vector calculus. Using eq.(2.76) and equating the terms with the same grade we have

$$\begin{aligned} &\nabla.\vec{E} = \rho\,; \quad \nabla\times\vec{B} - \partial_0\vec{E} = \vec{J}\,; \\ &\nabla\times\vec{E} + \partial_0\vec{B} = 0\,; \quad \nabla.\vec{B} = 0\,; \end{aligned} \tag{2.80}$$

which are Maxwell equations in the usual vector notation.

We now introduce the concept of Hertz potential [14] which permits us to find nontrivial solutions of the free "vacuum" Maxwell equation

$$\partial F = 0 \tag{2.81}$$

once we know nontrivial solutions of the scalar wave equation,

$$\Box\Phi = (\partial^2/\partial t^2 - \nabla^2)\Phi = 0;\ \Phi \in \sec\textstyle\bigwedge^0(M) \subset \sec\mathcal{C}\ell(M)\,. \tag{2.82}$$

Let $A \in \sec\bigwedge^1(M) \subset \sec\mathcal{C}\ell(M)$ be the vector potential. We fix the Lorentz gauge, *i.e.*, $\partial.A = -\delta A = 0$ such that $F = \partial A = (\mathrm{d} - \delta)A = \mathrm{d}A$. We have the following

Theorem. *Let $\pi \in \sec\bigwedge^2(M) \subset \sec\mathcal{C}\ell(M)$ be the so called Hertz potential. If π satisfies the wave equation,* i.e., $\Box\pi = \partial^2\pi = (\mathrm{d} - \delta)(\mathrm{d} - \delta)\pi = -(\mathrm{d}\delta + \delta\mathrm{d})\pi = 0$ *and if we take $A = -\delta\pi$, then $F = \partial A$ satisfies the Maxwell equation $\partial F = 0$.*

Proof. $A = -\delta\pi$ implies that $\delta A = -\delta^2\pi = 0$ and $F = \partial A = \mathrm{d}A$. Then $\partial F = (\mathrm{d}-\delta)(\mathrm{d}-\delta)A = \delta\mathrm{d}(\delta\pi) = -\delta^2\mathrm{d}\pi = 0$, since $\delta\mathrm{d}\pi = -\mathrm{d}\delta\pi$ from $\partial^2\pi = 0$.

From the above we see that if $\Phi \in \sec\bigwedge^0(M) \subset \sec\mathcal{C}\ell(M)$ satisfies $\partial^2\Phi = 0$, then we can find a non trivial solution of $\partial F = 0$, using a Hertz potential given, *e.g.*, by

$$\pi = \Phi\gamma_1\gamma_2 \ . \tag{2.83}$$

In Section 4.3 this equation is used, *e.g.* to generate the superluminal electromagnetic X-wave.

We now express the Hertz potential and its relation with the $\vec{E}$ and $\vec{B}$ fields, in order for our reader to see more familiar formulas. We write π as sum of electric and magnetic parts, *i.e.*,

$$\begin{gathered}\pi = \vec{\pi}_e + \mathbf{i}\vec{\pi}_m \\ \vec{\pi}_e = -\pi^{0i}\vec{\sigma}_i,\ \vec{\pi}_m = -\pi^{23}\vec{\sigma}_1 + \pi^{13}\vec{\sigma}_2 - \pi^{12}\vec{\sigma}_3\end{gathered} \tag{2.84}$$

Then, since $A = \partial\pi$ we have

$$\begin{aligned} A &= \frac{1}{2}(\partial\pi - \pi\overleftarrow{\partial}) \\ A\gamma_0 &= -\partial_0\vec{\pi}_e + \nabla.\vec{\pi}_e - (\nabla\times\vec{\pi}_m)\end{aligned}$$

and since $A = A^\mu\gamma_\mu$ we also have

$$A^0 = \nabla.\vec{\pi}_e \ ; \qquad \vec{A} = A^i\vec{\sigma}_i = -\frac{\partial}{\partial x^0}\vec{\pi}_e - \nabla\times\vec{\pi}_m \ .$$

Since $\vec{E} = -\nabla A^0 - \dfrac{\partial}{\partial x^0}\vec{A}$, $\vec{B} = \nabla\times\vec{A}$, we obtain

$$\begin{aligned}&\vec{E} = -\partial_0(\nabla\times\vec{\pi}_m) + \nabla\times\nabla\times\vec{\pi}_e \ ; \\ &\vec{B} = \nabla\times(-\partial_0\vec{\pi}_e - \nabla\times\vec{\pi}_m) = -\partial_0(\nabla\times\vec{\pi}_e) - \nabla\times\nabla\times\vec{\pi}_m \ .\end{aligned}$$

We define $\vec{E}_e, \vec{B}_e, \vec{E}_m, \vec{B}_m$ by

$$\begin{aligned}&\vec{E}_e = \nabla\times\nabla\times\vec{\pi}_e\,; \qquad \vec{B}_e = -\partial_0(\nabla\times\vec{\pi}_e)\,; \\ &\vec{E}_m = -\partial_0(\nabla\times\vec{\pi}_m)\,; \quad \vec{B}_m = -\nabla\times\nabla\times\vec{\pi}_m\,.\end{aligned} \tag{2.85}$$

We now introduce the 1-forms of stress-energy. Since $\partial F = 0$ we have $\widetilde{F}\widetilde{\partial} = 0$. Multiplying the first equation on the left by $\widetilde{F}$ and the second on the right by $\widetilde{F}$ and summing we have:

$$1/2(\widetilde{F}\partial F + \widetilde{F}\widetilde{\partial}F) = \partial_\mu((1/2)\widetilde{F}\gamma^\mu F) = \partial_\mu T^\mu = 0, \tag{2.86}$$

where $\widetilde{F}\widetilde{\partial} \equiv -(\partial_\mu\frac{1}{2}F_{\alpha\beta}\gamma^\alpha\gamma^\beta)\gamma^\mu$.

Now,

$$-\frac{1}{2}(F\gamma^\mu F)\gamma^\nu = -\frac{1}{2}(F\gamma^\mu F\gamma^\nu\rangle \tag{2.87}$$

Since $\gamma^\mu.F = \dfrac{1}{2}(\gamma^\mu F - F\gamma^\mu) = F.\gamma^\mu$, we have

$$\begin{aligned} T^{\mu\nu} &= -\langle(F.\gamma^\mu)F\gamma^\nu\rangle_0 - \frac{1}{2}\langle\gamma^\mu F^2\gamma^\nu\rangle_0 \\ &= -(F.\gamma^\mu).(F.\gamma^\nu) - \frac{1}{2}(F.F)\gamma^\mu.\gamma^\nu \\ &= F^{\mu\alpha}F^{\lambda\nu}\eta_{\alpha\lambda} + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta},\end{aligned} \tag{2.88}$$

which we recognize as the stress-energy momentum tensor of the electromagnetic field, and $T^\mu = T^{\mu\nu}\gamma_\nu$.

By writing $F = \vec{E} + \mathbf{i}\vec{B}$ as before we can immediately verify that

$$\begin{aligned} T_0 &= -\frac{1}{2}F\gamma_0 F \\ &= \left[\frac{1}{2}(\vec{E}^2 + \vec{B}^2) + (\vec{E}\times\vec{B})\right]\gamma_0. \end{aligned} \tag{2.89}$$

We have already shown that $\partial_\mu T^\mu = 0$, and we can easily show that

$$\partial . T^\mu = 0\,. \tag{2.90}$$

We now define the density of *angular momentum*. Choose as before a Lorentzian chart $\langle x^\mu\rangle$ of the maximal atlas of M and consider the 1-form $x = x^\mu\gamma_\mu = x_\mu\gamma^\mu$. Define

$$M_\mu = x\wedge T_\mu = \frac{1}{2}(x_\alpha T_{\mu\nu} - x_\nu T_{\alpha\mu})\gamma^\alpha\wedge\gamma^\nu\ .$$

It is trivial to verify that as $T_{\mu\nu} = T_{\nu\mu}$ and $\partial_\mu T^{\mu\nu} = 0$, it holds

$$\partial^\mu M_\mu = 0. \tag{2.91}$$

The *invariants* of the electromagnetic field F are $F.F$, $F\wedge F$ and $F^2 = F.F + F\wedge F$ with

$$F.F = -\frac{1}{2}F^{\mu\nu}F_{\mu\nu}\ ; \qquad F\wedge F = -\gamma_5 F^{\mu\nu}F^{\alpha\beta}\varepsilon_{\mu\nu\alpha\beta}\ . \tag{2.92}$$

Writing as before $F = \vec{E} + \mathbf{i}\vec{B}$ we have

$$F^2 = (\vec{E}^2 - \vec{B}^2) + 2\mathbf{i}\vec{E}.\vec{B} = F.F + F\wedge F. \tag{2.93}$$

2.6. Dirac theory in $\mathcal{C}\ell(M)$

Let $\Sigma = \{\gamma^\mu\} \in \sec\bigwedge^1(M) \subset \sec\mathcal{C}\ell(M)$ be an orthonormal basis. Let $\psi_\Sigma \in \sec(\bigwedge^0(M) + \bigwedge^2(M) + \bigwedge^4(M)) \subset \sec\mathcal{C}\ell(M)$ be the representative of a Dirac-Hestenes Spinor field in the basis Σ. Then the representative of Dirac equation in $\mathcal{C}\ell(M)$ is the following equation ($\hbar = c = 1$):

$$\partial\psi_\Sigma\gamma_1\gamma_2 + m\psi_\Sigma\gamma_0 = 0\ . \tag{2.94}$$

To see that, consider the *complexification* $\mathcal{C}\ell_C(M)$ of $\mathcal{C}\ell(M)$ called the *complex Clifford bundle*. Then $\mathcal{C}\ell_C(M) = \mathbb{C}\otimes\mathcal{C}\ell(M)$ and by the results of Section 2.4 we know that the typical fiber of $\mathcal{C}\ell_C(M)$ is $\mathcal{C}\ell_{4,1} = \mathbb{C}\otimes\mathcal{C}\ell_{1,3}$, the Dirac algebra.

Now let $\{\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4\} \subset \sec\bigwedge^1(M) \subset \sec\mathcal{C}\ell_C(M)$ be an orthonormal basis with

$$\begin{aligned} &\Gamma_a\Gamma_b + \Gamma_b\Gamma_a = 2g_{ab}\ , \\ &g_{ab} = diag(+1,+1,+1,+1,-1)\ . \end{aligned} \tag{2.95}$$

Let us identify $\gamma_\mu = \Gamma_\mu\Gamma_4$ and call $I = \Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4$. Since $I^2 = -1$ and I commutes with all elements of $\mathcal{C}\ell_{4,1}$ we identify I with $\mathrm{i} = \sqrt{-1}$ and γ_μ with the fundamental set of $\mathcal{C}\ell(M)$. Then if $\mathcal{A} \in \sec\mathcal{C}\ell_C(M)$ we have

$$\mathcal{A} = \Phi_s + A^\mu_C\gamma_\mu + \frac{1}{2}B^{\mu\nu}_C\gamma_\mu\gamma_\nu + \frac{1}{3!}\tau^{\mu\nu\rho}_C\gamma_\mu\gamma_\nu\gamma_\nu + \Phi_p\gamma_5, \tag{2.96}$$

where $\Phi_s, \Phi_p, A^\mu_C, B^{\mu\nu}_C, \tau^{\mu\nu\rho}_C \in \sec \mathbb{C} \otimes \bigwedge^0(M) \subset \sec \mathcal{C}\ell_C(M)$, *i.e.*, $\forall x \in M$, $\Phi_s(x)$, $\Phi_p(x)$, $A^\mu_C(x)$, $B^{\mu\nu}_C(x)$, $\tau^{\mu\nu\rho}_C(x)$ are complex numbers.

Now,

$$f = \frac{1}{2}(1+\gamma_0)\frac{1}{2}(1+\mathrm{i}\gamma_1\gamma_2)$$

is a primitive idempotent field of $\mathcal{C}\ell_C(M)$. We recall that, by eq.(2.51), $\mathrm{i}f = \gamma_2\gamma_1 f$. From (2.91) we can write the following equation in $\mathcal{C}\ell_C(M)$:

$$\begin{aligned} &\partial\psi_\Sigma\gamma_1\gamma_2 f + m\psi_\Sigma\gamma_0 f = 0 \\ &\partial\psi_\Sigma \mathrm{i} f - m\psi_\Sigma f = 0 \end{aligned}$$

and we have the following equation for $\Psi = \psi_\Sigma f$:

$$\mathrm{i}\partial\Psi - m\Psi = 0. \tag{2.97}$$

By eq.(2.49) and using for γ_μ the matrix representation eq.(2.47) (denoted here by $\underline{\gamma}_\mu$) we get that the matrix representation of eq.(2.94) is

$$\mathrm{i}\underline{\gamma}^\mu\partial_\mu|\Psi\rangle - m|\Psi\rangle = 0 \tag{2.98}$$

where now $|\Phi\rangle$ is a usual Dirac spinor field.

We now define a *potential* for the Dirac-Hestenes field ψ_Σ. Since $\psi_\Sigma \in \sec\mathcal{C}\ell^+(M)$ it is clear that there exist A, $B \in \sec\bigwedge^1(M) \subset \sec\mathcal{C}\ell(M)$ such that

$$\psi_\Sigma = \partial(A + \gamma_5 B), \tag{2.99}$$

since

$$\begin{aligned} \partial(A+\gamma_5 B) &= \partial.A + \partial\wedge A - \gamma_5\partial.B - \gamma_5\partial\wedge B \\ &= S + B + \gamma_5 P\,; \end{aligned}$$

$$S = \partial.A; \quad B = \partial\wedge A - \gamma_5\partial\wedge B; \quad P = -\partial.B\,.$$

We see that when $m = 0$, ψ_Σ satisfies the *Weyl equation*[9]

$$\partial\psi_\Sigma = 0\,. \tag{2.100}$$

Using eq.(2.100) we see that

$$\partial^2 A = \partial^2 B = 0. \tag{2.101}$$

This last equation allows us to find UPWs solutions of arbitrary speeds for the Weyl equation once we know UPWs solutions of the scalar wave equation $\square\Phi = 0$, $\Phi \in \sec\bigwedge^0(M) \subset \sec\mathcal{C}\ell(M)$. Indeed it is enough to put $\mathcal{A} = (A+\gamma_5 B) = \Phi(1+\gamma_5)v$, where v is a constant 1-form field. This result has been used in [15] to present subluminal and superluminal solutions of the Weyl equation. An example of a subluminal solution (indeed a stationary one) of the massless Dirac equation is obtained with the use of the "superpotential" $\mathcal{A}_0$:

$$\mathcal{A}_0 = \frac{C}{r}(\sin\Omega r\cos\Omega t\gamma^0 - \sin\Omega r\sin\Omega t\gamma^1\gamma^2\gamma^3). \tag{2.102}$$

[9] We recall again that a Weyl spinor must satisfy $\gamma_5\psi = \pm\psi\gamma_{21}$ (see *e.g.* [10]).

We have then

$$\begin{aligned}\psi_0 = \partial\mathcal{A}_0 &= \frac{C}{r^3}[-\Omega r^2 \sin\Omega r \sin\Omega t \\ &+ \gamma^0\gamma^1\lambda x\cos\Omega t + \gamma^0\gamma^2\lambda y\cos\Omega t \\ &+ \gamma^0\gamma^3\lambda z\cos\Omega t - \gamma^1\gamma^2\lambda z\sin\Omega t \\ &+ \gamma^1\gamma^3\lambda y\sin\Omega t - \gamma^2\gamma^3\lambda x\sin\Omega t \\ &+ \gamma^0\gamma^1\gamma^2\gamma^3\Omega r^2\sin\Omega r\cos\Omega t], \end{aligned} \tag{2.103}$$

where $\lambda = \Omega r\cos\Omega r - \sin\Omega r$, $r = \sqrt{x^2+y^2+z^2}$.

The above solution in the usual formalism reads

$$\Psi_0 = \begin{pmatrix} \mathrm{i}\sin\Omega t\left(\dfrac{\lambda z}{r^3} + \mathrm{i}\dfrac{\Omega}{r}\sin\Omega r\right) \\ \mathrm{i}\sin\Omega t\left(\dfrac{x+\mathrm{i}y}{r^3}\right)\lambda \\ -\cos\Omega t\left(\dfrac{\lambda z}{r^3} + \mathrm{i}\dfrac{\Omega}{r}\sin\Omega r\right) \\ -\cos\Omega t\left(\dfrac{x+\mathrm{i}y}{r^3}\right)\lambda \end{pmatrix}. \tag{2.104}$$

Another very interesting possibility for constructing solutions of Weyl equation is the following. Suppose that $F \in \sec\bigwedge^2(M) \subset \sec\mathcal{C}\ell(M)$, $F^2 \neq 0$, is a solution of $\partial F = 0$. Then $\psi = \mathrm{e}^F$ is a massless Dirac spinor field satisfying $\partial\psi = 0$.[10] Using this result and eq.(2.61) we can construct solutions of $\partial\psi_W = 0$. To end this Section we show how to construct luminal or superluminal solutions of the Dirac equation.

We know (see Section 3) that the Klein-Gordon equation has besides the subluminal solutions also luminal and superluminal solutions. Let Φ be a subluminal, luminal or superluminal solution of $\Box\Phi + m\Phi = 0$. Suppose Φ is a section of $\mathcal{C}\ell_C(M)$. Then in $\mathcal{C}\ell_C(M)$ we have the following factorization:

$$(\partial + \mathrm{i}m)(\partial - \mathrm{i}m)\Phi = 0. \tag{2.105}$$

Now

$$\Psi = (\partial - \mathrm{i}m)\Phi f \tag{2.106}$$

is a Dirac spinor field in $\mathcal{C}\ell_C(M)$, since

$$(\partial + \mathrm{i}m)\Psi = 0\,. \tag{2.107}$$

Ψ is then a subluminal, luminal or superluminal UPW solution of Dirac equation, depending on Φ.

3. EXTRAORDINARY SOLUTIONS OF THE (SCALAR) HOMOGENEOUS WAVE EQUATION AND OF KLEIN-GORDON EQUATION

[10] A closed expression for e^F is given in [16]. In particular, writing $F = \vec{E} + \mathbf{i}\vec{B} = z\hat{F}$ (eq.(2.77)), where $\hat{F} = z^*F/|z|^2$, $\hat{F}^2 = 1$, $z \in \mathbb{C}$, and $\hat{F}$ is a complex vector (in the Pauli algebra sense), then $\mathrm{e}^F = \mathrm{e}^{z\hat{F}} = \cosh z + \hat{F}\sinh z$.

3.1. Subluminal and superluminal solutions of the HWE

Consider the HWE ($c = 1$)

$$\frac{\partial^2}{\partial t^2}\Phi - \nabla^2\Phi = 0\ . \tag{3.1}$$

We now present some subluminal and superluminal solutions of this equation.

Subluminal and Superluminal Spherical Bessel Beams. To introduce these beams we define the variables

$$\xi_< = [x^2 + y^2 + \gamma_<^2(z - v_< t)^2]^{1/2}\ ; \tag{3.2}$$

$$\gamma_< = \frac{1}{\sqrt{1 - v_<^2}}\ ;\ \ \omega_<^2 - k_<^2 = \Omega_<^2\ ;\ \ v_< = \frac{\mathrm{d}\omega_<}{\mathrm{d}k_<}\ ; \tag{3.3}$$

$$\xi_> = [-x^2 - y^2 + \gamma_>^2(z - v_> t)^2]^{1/2}\ ; \tag{3.4}$$

$$\gamma_> = \frac{1}{\sqrt{v_>^2 - 1}}\ ;\ \ \omega_>^2 - k_>^2 = -\Omega_>^2\ ;\ \ v_> = \mathrm{d}\omega_>/\mathrm{d}k_>\ . \tag{3.5}$$

We can now easily verify that the functions $\Phi_<^{\ell m}$ and $\Phi_>^{\ell m}$ below are respectively subluminal and superluminal solutions of the HWE (see example 3 below for how to obtain these solutions). We have

$$\Phi_p^{\ell m}(t, \vec{x}) = C_\ell j_\ell(\Omega_p \xi_p) P_m^\ell(\cos\theta)\mathrm{e}^{im\theta}\mathrm{e}^{\mathrm{i}(\omega_p t - k_p z)} \tag{3.6}$$

where the index $p =<,\ >$, C_ℓ are constants, j_ℓ are the spherical Bessel functions, P_m^ℓ are the Legendre functions and (r, θ, φ) are the usual spherical coordinates. $\Phi_<^{\ell m}$ $[\Phi_>^{\ell m}]$ has phase velocity $(w_</k_<) < 1$ $[(w_>/k_>) > 1]$ and the modulation function $j_\ell(\Omega_< \xi_<)$ $[j_\ell(\Omega_> \xi_>)]$ moves with group velocity $v_<$ $[v_>]$, where $0 \leq v_< < 1$ $[1 < v_> < \infty]$. Both $\Phi_<^{\ell m}$ and $\Phi_>^{\ell m}$ are *undistorted progressive waves* (UPWs). This term has been introduced by Courant and Hilbert [4]. However, they didn't suspect of UPWs moving with speeds greater than $c = 1$. For use in the main text we write the explicit form of $\Phi_<^{00}$ and $\Phi_>^{00}$, which we denote simply by $\Phi_<$ and $\Phi_>$:

$$\Phi_p(t, \vec{x}) = C\frac{\sin(\Omega_p \xi_p)}{\xi_p}\mathrm{e}^{\mathrm{i}(\omega_p t - k_p z)}\ \ ;\ \ p =<\ \text{or}\ >\ . \tag{3.7}$$

When $v_< = 0$, we have $\Phi_< \to \Phi_0$,

$$\Phi_0(t, \vec{x}) = C\frac{\sin\Omega_< r}{r}\mathrm{e}^{\mathrm{i}\Omega_< t},\ r = (x^2 + y^2 + z^2)^{1/2}\ . \tag{3.8}$$

This solution has been found by Bateman in 1915 [17]. The superluminal solution $\Phi_>$ was discovered by Barut and Chandola in 1993 [18]. In what follows we show methods to obtain the solutions $\Phi_<$ and $\Phi_>$ for the HWE. When $v_> = \infty$, $\omega_> = 0$ and $\Phi_>^0 \to \Phi_\infty$,

$$\Phi_\infty(t, \vec{x}) = C_\infty\frac{\sinh\rho}{\rho}\mathrm{e}^{i\Omega_> z}\ ,\ \ \rho = (x^2 + y^2)^{1/2}\ . \tag{3.9}$$

We observe that if our interpretation of phase and group velocities is correct, then there must be a Lorentz frame where Φ_p is at rest. It is trivial to verify that in the coordinate chart $\langle x'^\mu\rangle$ which is a (nacs/I') (see Section 6 for more details), where $I' = (1 - v_<^2)^{-1/2}\partial/\partial t + (v_</\sqrt{1 - v_<^2})\partial/\partial z$ is a Lorentz frame moving with speed $v_<$ in the z direction relative to $I = \partial/\partial t$, Φ_p goes in $\Phi_0(t', \vec{x}')$ given by eq.(3.8) with $t \mapsto t'$,

$\vec{x} \mapsto \vec{x}'$. We can also verify that there is no Lorentz frame with velocity parameter $0 < v < 1$ where $\Phi_>$ is at rest.

Subluminal and Superluminal Bessel Beams. The solutions of the HWE in cylindrical coordinates are well known [14]. Here we recall how these solutions are obtained in order to present new subluminal and superluminal solutions of the HWE. In what follows the cylindrical coordinate functions are denoted by (ρ, θ, z), $\rho = (x^2 + y^2)^{1/2}$, $x = \rho \cos\theta$, $y = \rho \sin\theta$. We write for Φ:

$$\Phi(t, \rho, \theta, z) = f_1(\rho) f_2(\theta) f_3(t, z) \,. \tag{3.10}$$

Inserting (3.10) in (3.1) gives

$$\rho^2 \frac{\mathrm{d}^2}{\mathrm{d}\rho^2} f_1 + \rho \frac{\mathrm{d}}{\mathrm{d}\rho} f_1 + (B\rho^2 - \nu^2) f_1 = 0 \tag{3.11}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} + \nu^2 \right) f_2 = 0; \tag{3.12}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \frac{\partial^2}{\partial z^2} + B \right) f_3 = 0. \tag{3.13}$$

B and ν are separation constants. Since we want Φ to be periodic in θ we choose $\nu = n$ an integer. For B we consider two cases: (i) *Subluminal Bessel solution*, $B = \Omega_<^2 > 0$

In this case (3.11) is a Bessel equation and we have

$$\Phi_{J_n}^<(t, \rho, \theta, z) = C_n J_n(\rho \Omega_<) \mathrm{e}^{\mathrm{i}(k_< z - w_< t + n\theta)}, \quad n = 0, 1, 2, \ldots, \tag{3.14}$$

where C_n is a constant, J_n is the n-th order Bessel function and

$$\omega_<^2 - k_<^2 = \Omega_<^2 \,. \tag{3.15}$$

In [19] the $\Phi_{J_n}^<$ are called the n-th order non-diffracting Bessel beams.[11] Bessel beams are examples of undistorted progressive waves (UPWs). They are "subluminal" waves. Indeed, the group velocity for each wave is

$$v_< = \mathrm{d}\omega_< / \mathrm{d}k_<, \ 0 < v_< < 1 \,, \tag{3.16}$$

but the phase velocity of the wave is $(\omega_< / k_<) > 1$. That this interpretation is correct follows from the results of the acoustic experiment described in [5, 6].

It is convenient for what follows to define the variable η, called the axicon angle [20],

$$k_< = \overline{k}_< \cos\eta \,, \quad \Omega_< = \overline{k}_< \sin\eta \,, \quad 0 < \eta < \pi/2 \,. \tag{3.17}$$

Then

$$\overline{k}_< = \omega_< > 0 \tag{3.18}$$

and eq.(3.14) can be rewritten as $\Phi_{A_n}^< \equiv \Phi_{J_n}^<$, with

$$\Phi_{A_n}^< = C_n J_n(\overline{k}_< \rho \sin\eta) \mathrm{e}^{\mathrm{i}(\overline{k}_< z \cos\eta - \omega_< t + n\theta)}. \tag{3.19}$$

[11] The only difference is that $k_<$ is denoted by $\beta = \sqrt{\omega_<^2 - \Omega_<^2}$ and $\omega_<$ is denoted by $k' = \omega/c > 0$. (We use units where $c = 1$.)

In this form the solution is called in [19] the n-th order non-diffracting portion of the *Axicon Beam*.

Now, the phase velocity $v^{ph} = 1/\cos\eta$ is independent of $\overline{k}_<$, but, of course, it is dependent on $k_<$. We shall show below that waves constructed from the $\Phi^<_{J_n}$ beams can be *subluminal* or *superluminal* !

(ii) *Superluminal (Modified) Bessel Solution*, $B = -\Omega^2_> < 0$

In this case (3.11) is the modified Bessel equation and we denote the solutions by

$$\Phi^>_{K_n}(t,\rho,\theta,z) = C_n K_n(\Omega_>\rho)\mathrm{e}^{\mathrm{i}(k_> z - \omega_> t + n\theta)},\ n = 0,1,\ldots, \tag{3.20}$$

where K_n are the modified Bessel functions, C_n are constants and

$$\omega^2_> - k^2_> = -\Omega^2_> \ . \tag{3.21}$$

We see that $\Phi^>_{K_n}$ are also examples of UPWs, each of which has group velocity $v_> = \mathrm{d}\omega_>/\mathrm{d}k_>$ such that $1 < v_> < \infty$ and phase velocity $(\omega_>/k_>) < 1$. As in the case of the spherical Bessel beam (eq.(3.7)) we see again that our interpretation of phase and group velocities is correct. Indeed, for the superluminal (modified) Bessel beam there is no Lorentz frame where the wave is stationary.

The $\Phi^>_{K_0}$ beam was discussed by Band [21, 22] in 1988 as an example of superluminal motion. Band proposed to launch the $\Phi^>_{K_0}$ beam in the exterior of a cylinder of radius r_1 on which there is an appropriate superficial charge density. Since $K_0(\Omega_> r_1)$ is non singular, his solution works. In Section 4 we discuss some of Band's statements.

We are now prepared to present some other very interesting solutions of the HWE, in particular the so called X-waves, which are superluminal, as proved by the acoustical experiments described in [5, 6].

THEOREM (Lu and Greenleaf) *The three functions below are families of exact solutions of the HWE [eq.(3.1)] in cylindrical coordinates:*

$$\Phi_\eta(s) = \int_0^\infty T(\overline{k}_<)\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} A(\phi)f(s)\mathrm{d}\phi\right]\mathrm{d}\overline{k}_< \ ; \tag{3.22}$$

$$\Phi_K(s) = \int_{-\pi}^{\pi} D(\eta)\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} A(\phi)f(s)\mathrm{d}\phi\right]\mathrm{d}\eta \ ; \tag{3.23}$$

$$\Phi_L(\rho,\theta,z-t) = \Phi_1(\rho,\theta)\Phi_2(z-t) \ ; \tag{3.24}$$

where

$$s = \alpha_0(\overline{k}_<,\eta)\rho\cos(\theta-\phi) + b(\overline{k}_<,\eta)[z \pm c_1(\overline{k}_<,\eta)t] \tag{3.25}$$

and

$$c_1(\overline{k}_<,\eta) = \sqrt{1 + [\alpha_0(\overline{k}_<,\eta)/b(\overline{k}_<,\eta)]^2} \ . \tag{3.26}$$

In these formulas $T(\overline{k}_<)$ is any complex function (well behaved) of $\overline{k}_<$ and could include the *temporal frequency transfer function* of a radiator system, $A(\phi)$ is any complex function (well behaved) of ϕ and represents a *weighting function* of the integration with respect to ϕ, $f(s)$ is any complex function (well behaved) of s (solution of eq.(3.1)), $D(\eta)$ is any complex function (well behaved) of η and represents a weighting function of the integration with respect to η, called the axicon angle, $\alpha_0(\overline{k}_<,\eta)$ is any complex function of $\overline{k}_<$ and η, $b(\overline{k}_<,\eta)$ is any complex function of $\overline{k}_<$ and η.

As in the previous solutions, we take $c = 1$. Note that $\overline{k}_<$, η and the wave vector $k_<$ of the $f(s)$ solution of eq.(3.1) are related by eq.(3.17). Also $\Phi_2(z-t)$ is any complex function of $(z-t)$ and $\Phi_1(\rho,\theta)$ is any solution of the transverse Laplace equation, *i.e.*,

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}\right]\Phi_1(\rho,\theta)=0. \tag{3.27}$$

The proof is obtained by direct substitution of Φ_η, Φ_K and Φ_L in the HWE. Obviously, the exact solution Φ_L is an example of a luminal UPW, because if one "travels" with the speed $c = 1$, *i.e.*, with $z - t =$ constant, both the lateral and axial components, $\Phi_1(\rho,\theta)$ and $\Phi_2(z-t)$ will be the same for all time t and distance z. When $c_1(\overline{k},\eta)$ in eq.(3.25) is real, $(\pm)$ represent respectively backward and forward propagating waves.

We recall that $\Phi_\eta(s)$ and $\Phi_K(s)$ represent families of UPWs if $c_1(\overline{k}_<,\eta)$ is independent of $\overline{k}_<$ and η respectively. These waves travel to infinity at speed c_1. $\Phi_\eta(s)$ is a generalized function that contains some of the UPWs solutions of the HWE derived previously. In particular, if $T(\overline{k}_<) = \delta(\overline{k}_< - \overline{k}'_<)$, $\overline{k}'_< = \omega > 0$ is a constant and if $f(s) = \mathrm{e}^s$, $\alpha_0(\overline{k}_<,\eta) = -\mathrm{i}\Omega_<$, $b(\overline{k}_<,\eta) = \mathrm{i}\beta = \mathrm{i}\omega/c_1$, one obtains Durnin's UPW beam [21]

$$\Phi_{Durnin}(s)=\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}A(\phi)\mathrm{e}^{-\mathrm{i}\Omega_<\rho\cos(\theta-\phi)}\mathrm{d}\phi\right]\mathrm{e}^{\mathrm{i}(\beta z-\omega t)}. \tag{3.28}$$

If $A(\phi) = \mathrm{i}^n\mathrm{e}^{\mathrm{i}n\phi}$, we obtain the n-th order UPW Bessel beam $\Phi^<_{J_n}$ given by eq.(3.14). $\Phi^<_{A_n}(s)$ is obtained in the same way with the transformation $k_< = \overline{k}_<\cos\eta$; $\Omega_< = \overline{k}_<\sin\eta$.

The X-waves. We now present a superluminal UPW discovered in 1992 by Lu and Greenleaf [19] which, as discussed in [5, 20], is physically realizable in an approximate way in the acoustic case and can be used to generate Hertz potentials for the electromagnetic field (see Section 4). We take in eq.(3.22):

$$\begin{aligned}&T(\overline{k}_<)=B(\overline{k}_<)\mathrm{e}^{-a_0\overline{k}_<};\quad A(\phi)=\mathrm{i}^n\mathrm{e}^{\mathrm{i}n\phi};\quad f(s)=\mathrm{e}^s\\&\alpha_0(\overline{k}_<,\eta)=-\mathrm{i}\overline{k}_<\sin\eta\quad b(\overline{k}_<,\eta)=\mathrm{i}\overline{k}_<\cos\eta;\quad .\end{aligned} \tag{3.29}$$

Then we get

$$\Phi^>_{X_n}=\mathrm{e}^{\mathrm{i}n\theta}\int_0^\infty B(\overline{k}_<)J_n(\overline{k}_<\rho\sin\eta)\mathrm{e}^{-\overline{k}_<[a_0-\mathrm{i}(z\cos\eta-t)]}\mathrm{d}\overline{k}_<. \tag{3.30}$$

In eq.(3.30) $B(\overline{k}_<)$ is any well behaved complex function of $\overline{k}_<$ and represents a *transfer function of a practical radiator*, $\overline{k}_< = \omega$ and a_0 is a constant, and η is again called the axicon angle. Equation (3.30) shows that $\Phi^>_{X_n}$ is represented by a Laplace transform of the function $B(\overline{k}_<)J_n(\overline{k}_<\rho\sin\eta)$, and an azimuthal phase term $\mathrm{e}^{\mathrm{i}n\theta}$. The name *X-waves* for the $\Phi^>_{X_n}$ comes from the fact that these waves have an X-like shape in a plane containing the axis of symmetry of the waves [19, 5, 6].

The $\Phi^>_{XBB_n}$ waves. This wave is obtained from eq.(3.30) putting $B(\overline{k}_<) = a_0$. It is called the X-wave produced by an *infinite* aperture and *broad bandwidth*. We use in this case the notation $\Phi^>_{XBB_n}$. Under these conditions we get

$$\Phi^>_{XBB_n}=\frac{a_0(\rho\sin\eta)^n\mathrm{e}^{\mathrm{i}n\theta}}{\sqrt{M}(\tau+\sqrt{M})^n}\quad,\quad(n=0,1,2,\ldots) \tag{3.31}$$

where the subscript denotes "broadband". Also

$$M = (\rho \sin \eta)^2 + \tau^2 \; ; \tag{3.32}$$

$$\tau = [a_0 - \mathrm{i}(z \cos \eta - t)] \tag{3.33}$$

For $n = 0$ we get $\Phi^{>}_{XBB_0}$ and

$$\Phi^{>}_{XBB_0} = \frac{a_0}{\sqrt{(\rho \sin \eta)^2 + [a_0 - \mathrm{i}(z \cos \eta - t)]^2}} \, . \tag{3.34}$$

It is clear that all $\Phi^{>}_{XBB_n}$ are UPWs which propagate with speed $c_1 = 1/\cos \eta > 1$ in the z-direction. Our statement is justified for as can be easily seen (as in the modified superluminal Bessel beam) there is no Lorentz frame where $\Phi^{>}_{XBB_n}$ is at rest. Observe that this is the real speed of the wave; phase and group velocity concepts are not applicable here. Equation (3.34) does not give any dispersion relation. The $\Phi^{>}_{XBB_n}$ waves cannot be produced in practice as they have infinite energy (see Section 3.4), but a good approximation for them has been realized with *finite aperture* radiators [23, 5].

Recall that if in eq.(3.30) we put $B(\overline{k}_<)\mathrm{e}^{-a_0 \overline{k}_<} = \overline{A}(\overline{k}_<)$ and if we take into account that for each component Bessel beam in the packet the following dispersion relation holds:

$$\omega = k_< / \cos \eta \equiv k / \cos \eta \; (k_< = k) \, , \tag{3.35}$$

then for both the broad band X-waves (as *e.g.* eq.(3.34)) and the limited band X-waves where $A(k) \equiv \overline{A}(\overline{k}_<)$ is centered in k_0 we can write

$$\omega(k) = \omega(k_0) + \left.\frac{\mathrm{d}\omega}{\mathrm{d}k}\right|_{k_0} (k - k_0) = \omega_0 + \frac{\mathrm{d}\omega}{\mathrm{d}k_0}(k - k_0) \, , \tag{3.36}$$

where $\omega_0 = 1/\cos \eta$, $\mathrm{d}\omega/\mathrm{d}k_0 = 1/\cos \eta$. It follows that eq.(3.30) can be written

$$\begin{aligned} \Phi_{X_n}(t, x, y, z) \quad &= \quad \frac{\mathrm{e}^{\mathrm{i}n\theta}}{\cos \eta} \int \mathrm{d}k \, A(k) \, J_n(k\rho \tan \eta) \mathrm{e}^{\mathrm{i}(z - \frac{1}{\cos \eta} t)k} \\ &= \quad \Phi \left(0, x, y, z - \frac{1}{\cos \eta} t \right) \end{aligned} \tag{3.37}$$

showing that the X-waves propagate without distortion with speed $\mathrm{d}\omega/\mathrm{d}k_0 = 1/\cos \eta$.

We end this section with the commentary that in [6] we develop methods for projecting "finite aperture approximations" to the exact acoustic and electromagnetic solutions discussed in this paper.

3.2. Donnelly-Ziolkowski method for designing subluminal, luminal and superluminal UPW solutions of the HWE and the Klein-Gordon equation (KGE) [24, 25]

Consider first the HWE for Φ (eq.(3.1)) in a homogeneous medium. Let $\widetilde{\Phi}(\omega, \vec{k})$ be the Fourier transform of $\Phi(t, \vec{x})$, *i.e.*,

$$\widetilde{\Phi}(\omega, \vec{k}) = \int_{R^3} \mathrm{d}^3 x \int_{-\infty}^{+\infty} \mathrm{d}t \, \Phi(t, \vec{x}) \mathrm{e}^{-\mathrm{i}(\vec{k}\vec{x} - \omega t)}, \tag{3.38}$$

$$\Phi(t, \vec{x}) = \frac{1}{(2\pi)^4} \int_{R^3} \mathrm{d}^3 \vec{k} \int_{-\infty}^{+\infty} \mathrm{d}\omega \, \widetilde{\Phi}(\omega, \vec{k}) \mathrm{e}^{\mathrm{i}(\vec{k}\vec{x} - \omega t)}. \tag{3.39}$$

Inserting (3.38) in the HWE we get

$$(\omega^2 - \vec{k}^2)\widetilde{\Phi}(\omega, \vec{k}) = 0 \tag{3.40}$$

and we are going to look for solutions of the HWE and eq.(3.40) in the sense of distributions. We rewrite eq.(3.40) as

$$(\omega^2 - k_z^2 - \Omega^2)\widetilde{\Phi}(\omega, \vec{k}) = 0. \tag{3.41}$$

It is then obvious that any $\Phi(\omega, \vec{k})$ of the form

$$\widetilde{\Phi}(\omega, \vec{k}) = \Xi(\Omega, \beta)\, \delta[\omega - (\beta + \Omega^2/4\beta)]\, \delta[k_z - (\beta - \Omega^2/4\beta)]\ , \tag{3.42}$$

where $\Xi(\Omega, \beta)$ is an arbitrary weighting function, is a solution of eq.(3.41) since the δ-functions imply that

$$\omega^2 - k_z^2 = \Omega^2\ . \tag{3.43}$$

In 1985 Ziolkowski [26] found a *luminal* solution of the HWE called the Focus Wave Mode. To obtain this solution we choose, *e.g.*,

$$\Xi_{FWM}(\Omega, \beta) = \frac{\pi^2}{\mathrm{i}\beta} \exp(-\Omega^2 z_0/4\beta), \tag{3.44}$$

whence we get, assuming $\beta > 0$ and $z_0 > 0$,

$$\Phi_{FWM}(t, \vec{x}) = \mathrm{e}^{\mathrm{i}\beta(z+t)} \frac{\exp\{-\rho^2\beta/[z_0 + \mathrm{i}(z-t)]\}}{4\pi\mathrm{i}[z_0 + \mathrm{i}(z-t)]}. \tag{3.45}$$

Despite the velocities $v_1 = +1$ and $v_2 = -1$ appearing in the phase, the modulation function of Φ_{FWM} has very interesting properties, as discussed in details in [26]. It remains to observe that eq.(3.45) is a special case of Brittingham's formula [27].

Returning to eq.(3.42) we see that the δ-functions make any function of the Fourier transform variables ω, k_z and Ω to lie in a line on the surface $\omega^2 - k_z^2 - \Omega^2 = 0$ (eq.(3.41)). Then, the support of the δ-functions is the line

$$\omega = \beta + \Omega^2/4\beta;\ \ k_z = \beta - \Omega^2/4\beta\ . \tag{3.46}$$

The projection of this line in the (ω, k_z) plane is a straight line of slope -1 ending at the point (β, β). When $\beta = 0$ we must have $\Omega = 0$, and in this case the line is $\omega = k_z$ and $\Phi(t, \vec{x})$ is simply a superposition of plane waves, each one having frequency ω and traveling with speed $c = 1$ in the positive z direction.

Luminal UPWs solutions can be easily constructed by the ZM, but will not be discussed here. Instead, we now show how to use ZM to construct subluminal and superluminal solutions of the HWE.

First Example: Reconstruction of the subluminal Bessel Beams $\Phi^{<}_{J_0}$ and the superluminal $\Phi^{>}_{XBB_0}$ (X-wave). Starting from the "dispersion relation" $\omega^2 - k_z^2 - \Omega^2 = 0$, we define

$$\widetilde{\Phi}(\omega, \overline{k}) = \Xi(\overline{k}, \eta)\delta(k_z - \overline{k}\cos\eta)\delta(\omega - \overline{k}). \tag{3.47}$$

This implies that

$$k_z = \overline{k}\cos\eta; \quad \cos\eta = k_z/\omega, \ \ \omega > 0, \ \ -1 < \cos\eta < 1\,. \tag{3.48}$$

We take moreover

$$\Omega = \overline{k}\sin\eta; \quad \overline{k} > 0\ . \tag{3.49}$$

We recall that $\vec{\Omega} = (k_x, k_y)$, $\vec{\rho} = (x, y)$ and we choose $\vec{\Omega}.\vec{\rho} = \Omega\rho\cos\theta$. Now, putting eq.(3.47) in eq.(3.39) we get

$$\Phi(t,\vec{x}) = \frac{1}{(2\pi)^4}\int_0^\infty \mathrm{d}\overline{k}\,\overline{k}\sin^2\eta\left[\int_0^{2\pi}\mathrm{d}\theta\,\Xi(\overline{k},\eta)\mathrm{e}^{\mathrm{i}\overline{k}\rho\sin\eta\cos\theta}\right]\mathrm{e}^{\mathrm{i}(\overline{k}\cos\eta z - \overline{k}t)}. \tag{3.50}$$

Choosing

$$\Xi(\overline{k},\eta) = (2\pi)^3\,\frac{z_0\mathrm{e}^{-\overline{k}z_0\sin\eta}}{\overline{k}\sin\eta}, \tag{3.51}$$

where $z_0 > 0$ is a constant, we obtain

$$\Phi(t,\vec{x}) = z_0\sin\eta\int_0^\infty \mathrm{d}\overline{k}\,\mathrm{e}^{-\overline{k}z_0\sin\eta}\left[\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\theta\,\mathrm{e}^{\mathrm{i}\overline{k}\rho\sin\eta\cos\theta}\right]\mathrm{e}^{\mathrm{i}\overline{k}(\cos\eta\, z - t)}\ . \tag{3.52}$$

Calling $z_0\sin\eta = a_0 > 0$, the last equation becomes

$$\Phi^{>}_{X_0}(t,\vec{x}) = a_0\int_0^\infty \mathrm{d}\overline{k}\,\mathrm{e}^{-\overline{k}a_0}J_0(\overline{k}\rho\sin\eta)\mathrm{e}^{\mathrm{i}\overline{k}(\cos\eta\, z - t)}. \tag{3.53}$$

Writing $\overline{k} = \overline{k}_<$ and taking into account eq.(3.19) we see that

$$J_0(\overline{k}_<\rho\sin\eta)\mathrm{e}^{\mathrm{i}\overline{k}_<(z\cos\eta - t)} \tag{3.54}$$

is a subluminal Bessel beam, a solution of the HWE moving in the positive z direction. Moreover, a comparison of eq.(3.53) with eq.(3.30) shows that (3.53) is a particular superluminal X-wave, with $B(\overline{k}_<) = \mathrm{e}^{-a_0\overline{k}_<}$. In fact it is the $\Phi^{>}_{XBB_0}$ UPW given by eq.(3.34).

Second Example: Choosing in (3.50)

$$\Xi(\overline{k},\eta) = (2\pi)^3\mathrm{e}^{-z_0|\cos\eta|\overline{k}}\cot\eta \tag{3.55}$$

gives

$$\Phi^{>}(t,\vec{x}) = \cos^2\eta\int_0^\infty \mathrm{d}\overline{k}\,\overline{k}\mathrm{e}^{-z_0|\cos\eta|\overline{k}}J_0(\overline{k}\rho\sin\eta)\mathrm{e}^{-\mathrm{i}\overline{k}(\cos\eta z - t)} \tag{3.56}$$

$$= \frac{[z_0 - \mathrm{i}\,\mathrm{sgn}(\cos\eta)(z - t/\cos\eta)]}{[\rho^2\tan^2\eta + [z_0 + \mathrm{i}\,\mathrm{sgn}(\cos\eta)(z - t/\cos\eta)]^2]^{3/2}}. \tag{3.57}$$

Comparing eq.(3.56) with eq.(3.30) we discover that ZM produced in this example a more general $\Phi^{>}_{X_0}$ wave where $B(\overline{k}_<) = \mathrm{e}^{-z_0|\cos\eta|\overline{k}_<}$. Obviously $\Phi^{>}(t,\vec{x})$ given by eq.(3.57) moves with superluminal speed $(1/\cos\eta)$ in the positive or negative z-direction depending on the sign of $\cos\eta$, denoted $\mathrm{sgn}(\cos\eta)$.

In both examples studied above we see that the projection of the supporting line of eq.(3.47) in the (ω, k_z) plane is the straight line $k_z/\omega = \cos\eta$, and $\cos\eta$ is its reciprocal slope. This line is inside the "light cone" in the (ω, k_z) plane.

Third Example: Consider two arbitrary lines with the same reciprocal slope that we denote by $v > 1$, both running between the lines $\omega = \pm k_z$ in the upper half plane $\omega > 0$ and each cutting the ω-axis at different values β_1 and β_2.

The two lines are projections of members of a family of HWE solution lines and each one can be represented as a portion of the straight lines (between the lines $\omega = \pm k_z$)

$$k_z = v(\omega - \beta_1),\ k_z = v(\omega - \beta_2). \tag{3.58}$$

It is clear that on the solution line of the HWE, Ω takes values from zero up to a maximum value that depends on v and β and then back to zero.

We see also that the maximum value of Ω, given by $\beta v/\sqrt{v^2-1}$, on any HWE solution line occurs for those values of ω and k_z where the corresponding projection lines cut the line $\omega = vk_z$. It is clear that there are two points on any HWE solution line with the same value of Ω in the interval

$$0 < \Omega < v\beta/\sqrt{v^2-1} = \Omega_0. \tag{3.59}$$

It follows that in this case the HWE solution line breaks into two segments, as is the case of the projection lines. We can then associate two different weighting functions, one for each segment. We write

$$\begin{aligned}\widetilde{\Phi}(\Omega,\omega,k_z) \;=\; & \Xi_1(\Omega,v,\beta)\delta\left[k_z - \frac{v[\beta + \sqrt{\beta^2 v^2 - \Omega^2(v^2-1)}]}{(v^2-1)}\right] \times \\ & \times\delta\left[\omega - \frac{[\beta v^2 + \sqrt{v^2\beta^2 - \Omega^2(v^2-1)}]}{(v^2-1)}\right] + \\ & + \Xi_2(\Omega,v,\beta)\delta\left\{k_z - \frac{v[\beta - \sqrt{\beta^2 v^2 - \Omega^2(v^2-1)}]}{(v^2-1)}\right\} \times \\ & \times\delta\left\{\omega - \frac{[\beta v^2 - \sqrt{v^2\beta^2 - \Omega^2(v^2-1)}]}{(v^2-1)}\right\}.\end{aligned} \tag{3.60}$$

Now, choosing

$$\Xi_1(\Omega,v,\beta) = \Xi_2(\Omega,v,\beta) = (2\pi)^3/2\sqrt{\Omega_0^2 - \Omega^2} \tag{3.61}$$

we get

$$\Phi_{v,\beta}(t,\rho,z) = \Omega_0 \exp\left(\frac{\mathrm{i}\beta v(z - vt)}{\sqrt{v^2-1}}\right)\int_0^\infty \mathrm{d}\chi\, \chi J_0(\Omega_0\rho\chi)\cos\left\{\frac{\Omega_0 v}{\sqrt{v^2-1}}\frac{(z-t/v)}{\sqrt{1-\chi^2}}\right\}. \tag{3.62}$$

Then

$$\Phi_{v,\beta}(t,\rho,z) = \exp\left[\mathrm{i}\beta\frac{v(z-vt)}{\sqrt{v^2-1}}\right]\frac{\sin\left\{\Omega_0\sqrt{\frac{v^2}{(v^2-1)}(z-t/v)^2+\rho^2}\right\}}{\left\{\Omega_0\sqrt{\frac{v^2}{(v^2-1)}(z-t/v)^2+\rho^2}\right\}}. \tag{3.63}$$

If we call $v_< = \dfrac{1}{v} < 1$ and take into account the value of Ω_0 given by eq.(3.59), we can write eq.(3.63) as

$$\Phi_{v_<}(t,\rho,z) = \frac{\sin(\Omega_0\xi_<)}{\xi_<}\mathrm{e}^{\mathrm{i}\Omega_0(z-vt)}\ ; \tag{3.64}$$

$$\xi_< = \left[x^2 + y^2 + \frac{1}{1 - v_<^2}(z - v_< t)^2\right]^{1/2} \quad ; \tag{3.65}$$

which we recognize as the subluminal spherical Bessel beam of Section 3.1 (eq.(3.7)).

3.3. Klein-Gordon equation (KGE)

We show here the existence of subluminal, luminal and superluminal UPW solutions of the KGE. We want to solve

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\Phi^{KG}(t, \vec{x}) = 0, \qquad m > 0, \tag{3.66}$$

with the Fourier transform method. We obtain for $\widetilde{\Phi}^{KG}(\omega, \vec{k})$ (a generalized function) the equation

$$\{\omega^2 - k_z^2 - (\Omega^2 + m^2)\}\widetilde{\Phi}^{KG}(\omega, \vec{k}) = 0. \tag{3.67}$$

As in the case of the HWE, any solution of the KGE will have a transform $\widetilde{\Phi}(\omega, \vec{k})$ such that its support line lies on the surface

$$\omega^2 - k_z^2 - (\Omega^2 + m^2) = 0\ . \tag{3.68}$$

From eq.(3.68), calling $\Omega^2 + m^2 = K^2$, we see that we are in a situation identical to the HWE for which we showed the existence of subluminal, superluminal and luminal solutions. We write down as examples one solution of each kind.

Subluminal UPW solution of the KGE. To obtain this solution it is enough to change in eq.(3.63) $\Omega_0 = v\beta/\sqrt{v^2 - 1} \to \Omega_0^{KG} = \left[\left(\frac{v\beta}{\sqrt{v^2 - 1}}\right)^2 - m^2\right]^{1/2}$. We have,

$$\Phi_<^{KG}(t, \rho, z) = \exp\left\{\frac{\mathrm{i}\beta v(z - vt)}{\sqrt{v^2 - 1}}\right\}\frac{\sin(\Omega_0^{KG}\xi_<)}{\xi_<}\ ; \tag{3.69}$$

$$\xi_< = \left[x^2 + y^2 + \frac{1}{1 - v_<^2}(z - v_< t)^2\right]^{1/2},\ v_< = 1/v. \tag{3.70}$$

Luminal UPW solution of the KGE. To obtain a solution of this type it is enough, as in eq.(3.42), to write

$$\widetilde{\Phi}^{KG} = \Xi(\Omega, \beta)\delta[k_z - (\Omega^2 + (m^2 - \beta^2)/2\beta)]\delta[\omega - (\Omega^2 + (m^2 + \beta^2)/2\beta)]\,. \tag{3.71}$$

Choosing

$$\Xi(\Omega, \beta) = \frac{(2\pi)^2}{\beta}\exp(-z_0\Omega^2/2\beta),\ z_0 > 0, \tag{3.72}$$

gives

$$\Phi_\beta^{KG}(t, \vec{x}) = \mathrm{e}^{\mathrm{i}z(m^2 - \beta^2)/2\beta}\mathrm{e}^{-\mathrm{i}t(m^2 + \beta)/2\beta}\frac{\exp\{-\rho^2\beta/2[z_0 - \mathrm{i}(z - t)]\}}{[z_0 - \mathrm{i}(z - t)]}. \tag{3.73}$$

Superluminal UPW solution of the KGE. To obtain a solution of this kind we introduce a parameter v such that $0 < v < 1$ and write for $\widetilde{\Phi}^{KG}$ in (3.67)

$$\begin{aligned}\widetilde{\Phi}^{KG}_{v,\beta}(\omega,\Omega,k_z) &= \Xi(\Omega,v,\beta)\delta\left[\omega-\frac{\left(-\beta v^2+\sqrt{(\Omega^2+m^2)(1-v^2)+v^2\beta^2}\right)}{1-v^2}\right]\times \\ &\times \quad \delta\left[k_z-\frac{v\left(-\beta+\sqrt{(\Omega^2+m^2)(1-v^2)+v^2\beta^2}\right)}{1-v^2}\right] . \qquad (3.74)\end{aligned}$$

Next we choose

$$\Xi(\Omega,v,\beta)=\frac{(2\pi)^3\exp(-z_0\sqrt{\Omega_0^2+\Omega^2})}{\sqrt{\Omega_0^2+\Omega^2}}, \qquad (3.75)$$

where $z_0 > 0$ is an arbitrary parameter, and where

$$\Omega_0^2=\frac{\beta^2 v^2}{1-v^2}+m^2 . \qquad (3.76)$$

Then introducing $v_> = 1/v > 1$ and $\gamma_> = \dfrac{1}{\sqrt{v_>^2-1}}$, we get

$$\Phi^{KG>}_{v,\beta}(t,\vec{x})=\mathrm{e}^{\frac{\mathrm{i}(\Omega_0^2-m^2)(z-vt)}{\beta v}}\frac{\exp\left\{-\Omega_0\sqrt{[z_0-\mathrm{i}\gamma_>(z-v_>t)]^2+x^2+y^2}\right\}}{\sqrt{[z_0-\mathrm{i}\gamma_>(z-v_>t)]^2+x^2+y^2}} , \qquad (3.77)$$

which is a superluminal UPW solution of the KGE moving with speed $v_>$ in the z direction. From eq.(3.77) it is an easy task to reproduce the superluminal spherical Bessel beam which is solution of the HWE.

3.4. On the energy of the UPWs and the velocity of transport of energy

Let $\Phi_r(t,\vec{x})$ be a real solution of the HWE. Then, as it is well known [28], the energy of the solution is given by

$$\varepsilon=\int\int\int_{I\!R^3}\mathrm{d}\mathbf{v}\left[\left(\frac{\partial\Phi_r}{\partial t}\right)^2-\Phi_r\nabla^2\Phi_r\right]+\lim_{R\to\infty}\int\int_{S(R)}\mathrm{d}S\,\Phi_r\vec{n}.\nabla\Phi_r , \qquad (3.78)$$

where $S(R)$ is the 2-sphere of radius R .

We can easily verify that the real or imaginary parts of all UPWs solutions of the HWE presented above have infinite energy. The question arises of how to project superluminal waves, solutions of the HWE, with finite energy. This can be done if we recall that all UPWs discussed above can be indexed by at least one parameter that here we call α. Then, calling $\Phi_\alpha(t,\vec{x})$ the real or imaginary parts of a given UPW solution we may form “packets” of these solutions as

$$\Phi(t,\vec{x})=\int\mathrm{d}\alpha\,F(\alpha)\Phi_\alpha(t,\vec{x}) \qquad (3.79)$$

We now may test for a given solution Φ_α and for weighting function $F(\alpha)$ if the integral in eq.(3.78) is convergent. We can explicitly show that for some (but not all) of the solutions showed above (subluminal, luminal and superluminal) that for weighting functions satisfying certain integrability conditions the energy ε results finite.

It is particularly important in this context to quote that the finite aperture approximations for all UPWs discussed in this paper have, of course, finite energy. For the case in which Φ given by eq.(3.79) is used to generate solutions for, *e.g.*, Maxwell of Dirac fields, the conditions for the energy of these fields to be finite will in general be different from the condition that gives for Φ a finite energy. This problem will be discussed with more details in another paper.

To finish we remark that for a scalar field satisfying

$$\left(\frac{1}{c_*^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\Phi = 0 \tag{3.80}$$

we have as is well known that the flux of momentum is given by

$$\vec{S} = \nabla\Phi\frac{\partial\Phi}{\partial t} \tag{3.81}$$

and

$$u = \frac{1}{2}\left[(\nabla\Phi)^2 + \frac{1}{c_*^2}\left(\frac{\partial\Phi}{\partial t}\right)^2\right]. \tag{3.82}$$

We can immediately verify that if the speed of transport of energy is defined as $v_\varepsilon = |\vec{S}|/u$ then $v_\varepsilon \leq c_*$. The acoustic experiments reported in [5, 6] show nevertheless that for the X-wave the energy travels with speed $c_s/\cos\eta$ $(c_* = c_s)$. We thus see that the usual definitions of magnitudes such as density of energy and momentum and the velocity of transport of energy demand a careful revision. (See in this context also the discussion of Section 4.4.)

4. SUBLUMINAL AND SUPERLUMINAL UPW SOLUTIONS OF MAXWELL EQUATIONS (ME)

In this Section we make full use of the Clifford bundle formalism (CBF) summarized in Section 2, but we translate all the main results into the standard vector formalism used by physicists. We start by reanalyzing in Section 4.1 the plane wave solutions (PWS) of ME with the CBF. We clarify some misconceptions and explain the fundamental role of the duality operator γ_5 and the meaning of $\mathrm{i} = \sqrt{-1}$ in standard formulations of electromagnetic theory. Next, in Section 4.2 we discuss subluminal UPWs solutions of ME and an unexpected relation between these solutions and the possible existence of purely electromagnetic particles (PEPs) envisaged by Einstein [29], Poincaré [30], Ehrenfest [31] and recently discussed by Waite, Barut and Zeni [32, 33]. In Section 4.3 we discuss the theory of superluminal electromagnetic X-waves (SEXWs). In [5, 6] we present simulations of the motions of the SEXWs and of their finite aperture approximations, which can eventually be launched by appropriate physical devices.

4.1. Plane wave solutions of Maxwell equations

We recall that ME in vacuum can be written as [eq.(2.75)]

$$\partial F = 0, \tag{4.1}$$

where $F \sec \bigwedge^2(M) \subset \sec \mathcal{C}\ell(M)$. The well known PWS of eq.(4.1) are obtained as follows. We write in a given Lorentzian chart $\langle x^\mu \rangle$ of the maximal atlas of M a PWS moving in the z-direction

$$F = f\mathrm{e}^{\gamma_5 kx} \tag{4.2}$$

$$k = k^\mu \gamma_\mu,\ k^1 = k^2 = 0,\ x = x^\mu \gamma_\mu, \tag{4.3}$$

where $k, x \in \sec \bigwedge^1(M) \subset \sec \mathcal{C}\ell(M)$ and where f is a constant 2-form. From eqs.(4.1) and (4.2) we obtain

$$kF = 0 \tag{4.4}$$

Multiplying eq.(4.4) by k we get

$$k^2 F = 0 \tag{4.5}$$

and since $k \in \sec \bigwedge^1(M) \subset \sec \mathcal{C}\ell(M)$ then

$$k^2 = 0 \ \leftrightarrow \ k_0 = \pm|\vec{k}| = k^3, \tag{4.6}$$

i.e., the propagation vector is light-like. Also

$$F^2 = F.F + F \wedge F = 0 \tag{4.7}$$

as can be easily seen by multiplying both members of eq.(4.4) by F and taking into account that $k \neq 0$. Eq(4.7) says that the field invariants are null.

It is interesting to understand the fundamental role of the volume element γ_5 (duality operator) in electromagnetic theory. In particular since $\mathrm{e}^{\gamma_5 kx} = \cos kx + \gamma_5 \sin kx$, we see that

$$F = f \cos kx + \gamma_5 f \sin kx. \tag{4.8}$$

Writing $F = \vec{E} + \mathbf{i}\vec{B}$, (see eq.(2.72)) with $\mathbf{i} \equiv \gamma_5$ and choosing $f = \vec{e}_1 + \mathbf{i}\vec{e}_2$, $\vec{e}_1.\vec{e}_2 = 0$, $\vec{e}_1$, $\vec{e}_2$ constant vectors in the Pauli subalgebra sense, eq.(4.8) becomes

$$\vec{E} + \mathbf{i}\vec{B} = \vec{e}_1 \cos kx - \vec{e}_2 \sin kx + \mathbf{i}(\vec{e}_1 \sin kx + \vec{e}_2 \cos kx). \tag{4.9}$$

This equation is important because it shows that we must take care with the $\mathrm{i} = \sqrt{-1}$ that appears in usual formulations of Maxwell theory using complex electric and magnetic fields. The $\mathrm{i} = \sqrt{-1}$ in many cases unfolds a secret that can only be known through eq.(4.9). From eq.(4.4) we can also easily show that $\vec{k}.\vec{E} = \vec{k}.\vec{B} = 0$, *i.e.*, PWS of ME are *transverse* waves.

We can rewrite eq.(4.4) as

$$k\gamma_0\gamma_0 F \gamma_0 = 0 \tag{4.10}$$

and since $k\gamma_0 = k_0 + \vec{k},\ \gamma_0 F \gamma_0 = -\vec{E} + \mathbf{i}\vec{B}$ we have

$$\vec{k} f = k_0 f. \tag{4.11}$$

Now, we recall that in $\mathcal{C}\ell^+(M)$ (where, as we said in Section 2, the typical fiber is isomorphic to the Pauli algebra $\mathcal{C}\ell_{3,0}$) we can introduce [34] the operator of space conjugation denoted by $*$ such that writing $f = \vec{e} + \mathbf{i}\vec{b}$ we have

$$f^* = -\vec{e} + \mathbf{i}\vec{b} \ \ ; \ \ k_0^* = k_0 \ \ ; \ \ \vec{k}^* = -\vec{k}. \tag{4.12}$$

We can now interpret the two solutions of $k^2 = 0$, *i.e.*, $k_0 = |\vec{k}|$ and $k_0 = -|\vec{k}|$ as corresponding to the solutions $k_0 f = \vec{k} f$ and $k_0 f^* = -\vec{k} f^*$; f and f^* correspond in quantum theory to "photons" which are of positive or negative helicities. We can interpret $k_0 = |\vec{k}|$ as a particle and $k_0 = -|\vec{k}|$ as an antiparticle.

Summarizing we have the following important facts concerning PWS of ME: (i) the propagation vector is light-like, $k^2 = 0$; (ii) the field invariants are null, $F^2 = 0$; (iii) the PWS are transverse waves, *i.e.*, $\vec{k}.\vec{E} = \vec{k}.\vec{B} = 0$.

4.2. Subluminal solutions of Maxwell equations and purely electromagnetic particles

We take $\Phi \in \sec(\bigwedge^0(M) \oplus \bigwedge^4(M)) \subset \sec \mathcal{C}\ell(M)$ and consider the following Hertz potential $\pi \in \sec \bigwedge^2(M) \subset \sec \mathcal{C}\ell(M)$ [eq.(2.83)]

$$\pi = \Phi\gamma^1\gamma^2. \tag{4.13}$$

We now write

$$\Phi(t, \vec{x}) = \phi(\vec{x})\mathrm{e}^{\gamma_5 \Omega t}. \tag{4.14}$$

Since π satisfies the wave equation, we have

$$\nabla^2\phi(\vec{x}) + \Omega^2\phi(\vec{x}) = 0 \tag{4.15}$$

Solutions of eq.(3.15) (the Helmholtz equation) are well known. Here we consider the simplest solution in spherical coordinates,

$$\phi(\vec{x}) = C\frac{\sin\Omega r}{r} \quad , \quad r = \sqrt{x^2 + y^2 + z^2}, \tag{4.16}$$

where C is an arbitrary real constant. From the results of Section 2 we obtain the following stationary electromagnetic field, which is at rest in the reference frame Z where $\langle x^\mu \rangle$ are naturally adapted coordinates to Z (see Section 5 for the definition of these concepts).

$$\begin{aligned}
F_0 = \frac{C}{r^3}&[\sin\Omega t(\alpha\Omega r \sin\theta\sin\varphi - \beta\cos\theta\sin\theta\cos\varphi)\gamma_0\gamma_1 \\
&- \sin\Omega t(\alpha\Omega r\sin\theta\cos\varphi + \beta\sin\theta\cos\theta\sin\varphi)\gamma_0\gamma_2 \\
&+ \sin\Omega t(\beta\sin^2\theta - 2\alpha)\gamma_0\gamma_3 + \cos\Omega t(\beta\sin^2\theta - 2\alpha)\gamma_1\gamma_2 \\
&+ \cos\Omega t(\beta\sin\theta\cos\theta\sin\varphi + \alpha\Omega r\sin\theta\cos\varphi)\gamma_1\gamma_3 \\
&+ \cos\Omega t(-\beta\sin\theta\cos\theta\cos\varphi + \alpha\Omega r\sin\theta\sin\varphi)\gamma_2\gamma_3]
\end{aligned} \tag{4.17}$$

with $\alpha = \Omega r\cos\Omega r - \sin\Omega r$ and $\beta = 3\alpha + \Omega^2 r^2 \sin\Omega r$. Observe that F_0 is regular at the origin and vanishes at infinity. Let us rewrite the solution using the Pauli-algebra in $\mathcal{C}\ell^+(M)$. Writing ($\mathbf{i} \equiv \gamma_5$)

$$F_0 = \vec{E}_0 + \mathbf{i}\vec{B}_0 \tag{4.18}$$

we get

$$\vec{E}_0 = \vec{W}\sin\Omega t \quad , \quad \vec{B}_0 = \vec{W}\cos\Omega t \tag{4.19}$$

with

$$\vec{W} = -C\left(\frac{\alpha\Omega y}{r^3} - \frac{\beta xz}{r^5}, -\frac{\alpha\Omega x}{r^3} - \frac{\beta yz}{r^5}, \frac{\beta(x^2+y^2)}{r^5} - \frac{2\alpha}{r^3}\right). \tag{4.20}$$

We verify that $\mathrm{div}\vec{W} = 0$, $\mathrm{div}\vec{E}_0 = \mathrm{div}\vec{B}_0 = 0$, $\mathrm{rot}\vec{E}_0 + \partial\vec{B}_0/\partial t = 0$, $\mathrm{rot}\vec{B}_0 - \partial\vec{E}_0/\partial t = 0$, and

$$\mathrm{rot}\vec{W} = \Omega\vec{W}. \tag{4.21}$$

Now, from eq.(2.89) we know that $T_0 = -\frac{1}{2}F\gamma_0 F$ is the 1-form representing the energy density and the Poynting vector. It follows that $\vec{E}_0 \times \vec{B}_0 = 0$, *i.e.*, the solution has zero angular momentum. The energy density $u = S^{00}$ is given by

$$u = \frac{1}{r^6}[\sin^2\theta(\Omega^2 r^2\alpha^2 + \cos^2\theta\beta^2) + (\beta\sin^2\theta - 2\alpha)^2]. \tag{4.22}$$

Then $\int\int\int_{I\!R^3} u\,\mathrm{d}\mathbf{v} = \infty$. As for the case of the scalar field (see Section 4.4) a finite energy solution can be constructed by considering "wave packets" with a distribution of intrinsic frequencies $F(\Omega)$ satisfying appropriate conditions. Many possibilities exist, but they will not be discussed here. Instead, we prefer to direct our attention to eq.(4.21). As it is well known, this is a very important equation (called the force free equation [32]) that appears *e.g.* in hydrodynamics and in several different situations in plasma physics [35]. The following considerations are more important.

Einstein [29] among others (see [32] for a review) studied the possibility of constructing PEPs. He started from Maxwell equations for a PEP configuration described by an electromagnetic field F_p and a current density J_p, where

$$\partial F_p = J_p \tag{4.23}$$

and rightly concluded that the condition for existence of PEPs is

$$J_p.F_p = 0. \tag{4.24}$$

This condition implies in vector notation

$$\rho_p \vec{E}_p = 0\ ,\quad \vec{\jmath}_p.\vec{E}_p = 0\ ,\quad \vec{\jmath}_p \times \vec{B}_p = 0 \tag{4.25}$$

From eq.(4.25) Einstein concluded that the only possible solution of eq.(4.23) with the subsidiary condition given by eq.(4.24) is $J_p = 0$. However, this conclusion is correct, as pointed in [32, 33], only if $J_p^2 > 0$, *i.e.*, if J_p is a time-like current density. If we suppose that J_p can be spacelike, *i.e.*, $J_p^2 < 0$, there exists a reference frame where $\rho_p = 0$ and a possible solution of eq.(3.24) is

$$\rho_p = 0\quad,\quad \vec{E}_p.\vec{B}_p = 0\quad,\quad \vec{\jmath}_p = KC\vec{B}_p, \tag{4.26}$$

where $K = \pm 1$ is called the chirality of the solution and C is a real constant. In [32, 33] static solutions of eq.(4.23) and (4.24) are exhibited where $\vec{E}_p = 0$. In this case we can verify that $\vec{B}_p$ satisfies

$$\nabla \times \vec{B}_p = KC\vec{B}_p. \tag{4.27}$$

Now, if we choose $F \in \sec \bigwedge^2(M) \subset \sec \mathcal{C}\ell(M)$ such that

$$\begin{gathered} F_0 = \vec{E}_0 + \mathbf{i}\vec{B}_0\,, \\ \vec{E}_0 = \vec{B}_p \cos\Omega t\,,\quad \vec{B}_0 = \vec{B}_p \sin\Omega t \end{gathered} \tag{4.28}$$

and $\Omega = KC > 0$, we immediately realize that

$$\partial F_0 = 0\,. \tag{4.29}$$

This is an amazing result, since it means that the free Maxwell equations may have stationary solutions that model PEPs. In such solutions the structure of the field F_0 is such that we can write

$$\begin{gathered} F_0 = F_p^{'} + \overline{F} = \mathbf{i}\vec{W}\cos\Omega t - \vec{W}\sin\Omega t\,, \\ \partial F_p^{'} = -\partial\overline{F} = J_p^{'}\,, \end{gathered} \tag{4.30}$$

i.e., $\partial F_0 = 0$ is equivalent to a field plus a current. This opens several interesting possibilities for modeling PEPs (see also [36]) and we discuss more this issue in another publication.

We observe that moving subluminal solutions of ME can be easily obtained choosing as Hertz potential, *e.g.*,

$$\pi^{<}(t,\vec{x}) = C\frac{\sin\Omega\xi_{<}}{\xi_{<}}\exp[\gamma_5(\omega_{<}t - k_{<}z)]\gamma_1\gamma_2\,, \tag{4.31}$$

$$\begin{aligned} &\omega_{<}^2 - k_{<}^2 = \Omega_{<}^2\,,\\ &\xi_{<} = [x^2 + y^2 + \gamma_{<}^2(z - v_{<}t)^2]\,, \\ &\gamma_{<} = \frac{1}{\sqrt{1 - v_{<}^2}}\,,\quad v_{<} = \mathrm{d}\omega_{<}/\mathrm{d}k_{<}\,. \end{aligned} \tag{4.32}$$

We are not going to write explicitly the expression for $F^{<}$ corresponding to $\pi^{<}$ because it is very long and will not be used in what follows.

We end this Section with the following observations: (i) In general for subluminal solutions of ME (SSME) the propagation vector satisfies an equation like eq.(4.30). (ii) As can be easily verified, for a SSME the field invariants are non-null. (iii) A SSME is not a transverse wave. This can be seen explicitly from eq.(4.20).

Conditions (i), (ii), (iii) are in contrast with the case of the PWS of ME. In [37, 38] Rodrigues and Vaz showed that for free electromagnetic fields ($\partial F = 0$) such that $F^2 \neq 0$, there exists a Dirac-Hestenes equation for $\psi \in \sec(\bigwedge^0(M)+\bigwedge^2(M)+\bigwedge^4(M)) \subset \sec\mathcal{C}\ell(M)$ where $F = \psi\gamma_1\gamma_2\widetilde{\psi}$. This was the reason why Rodrigues and Vaz discovered subluminal and superluminal solutions of Maxwell equations (and also of Weyl equation [15]) which solve the Dirac-Hestenes equation [eq.(2.94)]. An explicit superluminal solution of Maxwell equations is given in [15] using as Hertz potential $\Pi_{>} = \Phi_{>}\gamma^{12}$ where $\Phi_{>}$ is given by eq.(3.6).

4.3. The superluminal electromagnetic X-wave (SEXW)

In this Section we present a family of solutions of Maxwell equations called the superluminal electromagnetic X-waves F_{XBB_n}. A solution dual to F_{XBB_n}, called $\star F_{XBB_n}$ has been first presented by Lu and Greenleaf in an unpublished paper [39]. Later the solutions F_{XBB_n} and others associated with it have been studied in detail [40, 6].

To simplify the matter in what follows we now suppose that the functions Φ_{X_n} [eq.(3.30)] and Φ_{XBB_n} [eq.(3.31)] which are superluminal solutions of the scalar wave equation are 0-forms sections of the complexified Clifford bundle $\mathcal{C}\ell_C(M) = \mathbb{C}\otimes\mathcal{C}\ell(M)$ (see Section 2.6). We rewrite eqs.(3.30) and (3.34) as ($n = 0, 1, 2, \ldots$)

$$\Phi_{X_n}(t,\vec{x}) = \mathrm{e}^{in\theta}\int_0^\infty B(\overline{k})J_n(\overline{k}\rho\sin\eta)\mathrm{e}^{-\overline{k}[a_0 - \mathrm{i}(z\cos\eta - t)]}\mathrm{d}\overline{k} \tag{4.33}$$

and choosing $B(\overline{k}) = a_0$ we have

$$\Phi_{XBB_n}(t,\vec{x}) = \frac{a_0(\rho\sin\eta)^n\mathrm{e}^{in\theta}}{\sqrt{M}(\tau + \sqrt{M})^n} \tag{4.34}$$

$$M = (\rho\sin\eta)^2 + \tau^2 \quad ; \quad \tau = [a_0 - \mathrm{i}(z\cos\eta - t)]. \tag{4.35}$$

Further, we suppose now that the Hertz potential π, the vector potential A and the corresponding electromagnetic field F are appropriate sections of $\mathcal{C}\ell_C(M)$. We take

$$\pi = \Phi\gamma_1\gamma_2 \in \sec\mathbb{C}\otimes\bigwedge^2(M) \subset \sec\mathcal{C}\ell_C(M), \tag{4.36}$$

where Φ can be $\Phi_{X_n}, \Phi_{XBB_n}, \Phi_{XBL_n}$. Let us start by giving the explicit form of the F_{XBB_n} *i.e.*, the SEXWs. In this case eq.(2.84) gives $\pi = \mathbf{i}\,\vec{\pi}_m$ and

$$\vec{\pi}_m = \Phi_{XBB_n}\boldsymbol{z} \tag{4.37}$$

where $\boldsymbol{z}$ is the versor of the z-axis. Also, let $\boldsymbol{\rho}, \boldsymbol{\theta}$ be respectively the versors of the ρ and θ directions where (ρ, θ, z) are the usual cylindrical coordinates. Writing

$$F_{XBB_n} = \vec{E}_{XBB_n} + \gamma_5 \vec{B}_{XBB_n} \tag{4.38}$$

we obtain from equations (3.31) and (2.85):

$$\vec{E}_{XBB_n} = -\frac{\boldsymbol{\rho}}{\rho}\frac{\partial^2}{\partial t \partial \theta}\Phi_{XBB_n} + \boldsymbol{\theta}\frac{\partial^2}{\partial t \partial \rho}\Phi_{XBB_n} \tag{4.39}$$

$$\vec{B}_{XBB_n} = \boldsymbol{\rho}\frac{\partial^2}{\partial \rho \partial z}\Phi_{XBB_n} + \boldsymbol{\theta}\frac{1}{\rho}\frac{\partial^2}{\partial \theta \partial z}\Phi_{XBB_n} + \boldsymbol{z}\left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2}\right)\Phi_{XBB_n} \tag{4.40}$$

Explicitly we get for the components in cylindrical coordinates,

$$(\vec{E}_{XBB_n})_\rho = -\frac{1}{\rho} n \frac{M_3}{\sqrt{M}}\Phi_{XBB_n} \tag{4.41a}$$

$$(\vec{E}_{XBB_n})_\theta = \frac{1}{\rho}\mathrm{i}\frac{M_6}{\sqrt{M}M_2}\Phi_{XBB_n} \tag{4.41b}$$

$$(\vec{B}_{XBB_n})_\rho = \cos\eta (\vec{E}_{XBB_n})_\theta \tag{4.41c}$$

$$(\vec{B}_{XBB_n})_\theta = -\cos\eta (\vec{E}_{XBB_n})_\rho \tag{4.41d}$$

$$(\vec{B}_{XBB_n})_z = -\sin^2\eta \frac{M_7}{\sqrt{M}}\Phi_{XBB_n}. \tag{4.41e}$$

The functions $M_i (i = 2, \ldots, 7)$ in (4.41) are

$$M_2 = \tau + \sqrt{M} \tag{4.42a}$$

$$M_3 = n + \frac{1}{\sqrt{M}}\tau \tag{4.42b}$$

$$M_4 = 2n + \frac{3}{\sqrt{M}}\tau \tag{4.42c}$$

$$M_5 = \tau + n\sqrt{M} \tag{4.42d}$$

$$M_6 = (\rho^2 \sin^2\eta \frac{M_4}{M} - nM_3)M_2 + n\rho^2\frac{M_5}{M}\sin^2\eta \tag{4.42e}$$

$$M_7 = (n^2 - 1)\frac{1}{\sqrt{M}} + 3n\frac{1}{M}\tau + 3\frac{1}{\sqrt{M^3}}\tau^2 \tag{4.42f}$$

We immediately see from eqs.(4.41) that the F_{XBB_n} are indeed superluminal UPWs solutions of ME, propagating with speed $1/\cos\eta$ in the z-direction. That F_{XBB_n} are UPWs is trivial and that they propagate with speed $c_1 = 1/\cos\eta$ follows because F_{XBB_n} depends only on the combination of variables $(z - c_1 t)$ and any derivatives of Φ_{XBB_n} will keep the $(z - c_1 t)$ dependence structure.

Now, the Poynting vector $\vec{P}_{XBB_n}$ and the energy density u_{XBB_n} for F_{XBB_n} are obtained by considering the real parts of $\vec{E}_{XBB_n}$ and $\vec{B}_{XBB_n}$. We have,

$$(\vec{P}_{XBB_n})_\rho = -Re\{(\vec{E}_{XBB_n})_\theta\}Re\{(\vec{B}_{XBB_n})_z\} \tag{4.43a}$$

$$(\vec{P}_{XBB_n})_\theta = Re\{(\vec{E}_{XBB_n})_\rho\}Re\{(\vec{B}_{XBB_n})_z\} \tag{4.43b}$$

$$(\vec{P}_{XBB_n})_z = \cos\eta\left[|Re\{(\vec{E}_{XBB_n})_\rho\}|^2 + |Re\{(\vec{E}_{XBB_n})_\theta\}|^2\right] \tag{4.43c}$$

$$\begin{aligned} u_{XBB_n} &= (1+\cos^2\eta)\left[|Re\{(\vec{E}_{XBB_n})_\rho\}|^2 + |Re\{(\vec{E}_{XBB_n})_\theta\}|^2\right] \\ &+ |Re\{(\vec{B}_{XBB_n})_z\}|^2 \,. \end{aligned} \tag{4.44}$$

The total energy of F_{XBB_n} is then

$$\varepsilon_{XBB_n} = \int_{-\pi}^{\pi} \mathrm{d}\theta \int_{-\infty}^{+\infty} \mathrm{d}z \int_0^\infty \rho\,\mathrm{d}\rho\, u_{XBB_n} \tag{4.45}$$

Since as $z \to \infty$ $\vec{E}_{XBB_n}$ decrease as $1/|z - t\cos\eta|^{1/2}$ which occurs for the X-branches of F_{XBB_n}, ε_{XBB_n} may not be finite. Nevertheless, as in the case of the acoustic X-waves, which experiments have shown to travel with $v > c_s$ [6], we are quite sure that a finite aperture approximation to F_{XBL_n} (FAAF_{XBL_n}) can be launched over a large distance. Indeed in [6] computer simulations for the motion of FAAF_{XBB_n} are exhibited showing that with an antenna of 20 m of diameter a FAAF_{XBB_n} centered at a frequency of 700 GHz propagates with superluminal speed without appreciable distortion up to 100 Km. See also [40]. Obviously in this case the total energy of the FAAF_{XBL_n} is finite.

We conclude this Section observing that in general both subluminal and superluminal UPW solutions of ME have non-null field invariants and are not transverse waves. In particular our solutions have a longitudinal component along the z-axis. This result is important because it shows that, contrary to the speculations of Evans [41], we do not need an electromagnetic theory with a non zero photon-mass, *i.e.*, with F satisfying Proca's equation (as proposed also by de Broglie [42]) in order to have an electromagnetic wave with a longitudinal component. Since Evans presents evidence [41] of the existence of longitudinal magnetic fields in many different physical situations, we conclude that the theoretical and experimental study of subluminal and superluminal UPW solutions of ME must be continued.

4.4. The velocity of transport of energy of the UPW solutions of Maxwell equations

Since we found in this paper UPWs solutions of Maxwell equations with speeds $0 \leq v < \infty$, the following question arises naturally: Which is the velocity of transport of the energy of a superluminal UPW (or quasi UPW) solution of ME?

We can find in many physics textbooks (*e.g.* [43]) and in scientific papers [21, 22] the following argument. Consider an arbitrary solution of ME in vacuum $\partial F = 0$. Then if $F = \vec{E} + \mathbf{i}\vec{B}$ (see eq.(2.77)) it follows that the Poynting vector and the energy density of the field are

$$\vec{P} = \vec{E} \times \vec{B}\,, \quad u = \frac{1}{2}(\vec{E}^2 + \vec{B}^2)\,. \tag{4.46}$$

It is obvious that the following inequality always holds:

$$v_\varepsilon = \frac{|\vec{P}|}{u} \leq 1. \tag{4.47}$$

Now, the conservation of energy-momentum reads in integral form over a finite volume V with boundary $S = \partial V$:

$$\frac{\partial}{\partial t}\left\{\iiint_V \mathrm{d}\mathbf{v}\frac{1}{2}(\vec{E}^2 + \vec{B}^2)\right\} = \oint_S d\vec{S}.\vec{P} \tag{4.48}$$

Eq.(4.48) is interpreted saying that $\oint_S d\vec{S}.\vec{P}$ is the field energy flux across the surface $S = \partial V$, so that $\vec{P}$ is the flux density — the amount of field energy passing through a unit area of the surface in unit time. Now, for plane wave solutions of Maxwell equations,

$$v_\varepsilon = 1 \tag{4.49}$$

and this result gives origin to the "dogma" that free electromagnetic fields transport energy at speed $v_\varepsilon = c = 1$.

However $v_\varepsilon \leq 1$ is true even for subluminal and superluminal solutions of ME, as the ones discussed in Sections 4.2 and 4.3. The same is true for the superluminal modified Bessel beam found by Band [21] in 1987. There he claims that since $v_\varepsilon \leq 1$ there is no conflict between superluminal solutions of ME and Relativity Theory since what Relativity forbids is the propagation of energy with speed greater than c.

Here we challenge this conclusion. The fact is that as well known $\vec{P}$ is not uniquely defined. Eq.(4.48) continues to hold true if we substitute $\vec{P} \mapsto \vec{P} + \vec{P}'$ with $\nabla.\vec{P}' = 0$. But of course we can easily find for subluminal, luminal or superluminal solutions of Maxwell equations a $\vec{P}'$ such that

$$\frac{|\vec{P} + \vec{P}'|}{u} \geq 1. \tag{4.50}$$

We arrive at the conclusion that the question of the transport of energy in superluminal UPWs solutions of ME is an experimental question. For the acoustic superluminal X-wave solution of the HWE (see [5, 6]) the energy around the peak area flows together with the wave, *i.e.*, with speed $c_1 = c_s/\cos\eta$ while (as we said in Section 3.4) the usual theory predicts for the speed of propagation of sound waves that $|\vec{S}|/u < c_s$, where $\vec{S}$ is the flux of momentum and u is the energy density (eqs. (3.81) and (3.82)). Since we can see no possibility of the field energy of the superluminal electromagnetic wave to travel outside the wave we are confident to state that the velocity of energy transport of superluminal electromagnetic waves is superluminal.

Before ending we give another example to illustrate that eq.(4.47) is devoid of physical meaning. Consider a spherical conductor in electrostatic equilibrium with uniform superficial charge density (total charge Q) and with a dipole magnetic moment. Then we have

$$\vec{E} = Q\frac{\mathbf{r}}{r^2} \quad ; \quad \vec{B} = \frac{C}{r^3}(2\cos\theta\,\mathbf{r} + \sin\theta\,\boldsymbol{\theta}) \tag{4.51}$$

and

$$\vec{P} = \vec{E} \times \vec{B} = \frac{CQ}{r^5}\sin\theta\,\boldsymbol{\varphi} \quad , \quad u = \frac{1}{2}\left[\frac{Q^2}{r^4} + \frac{C^2}{r^6}(3\cos^2\theta + 1)\right]. \tag{4.52}$$

Thus

$$\frac{|\vec{P}|}{u} = \frac{2rCQ\sin\theta}{r^2Q^2 + C^2(3\cos^2\theta + 1)} \neq 0, \quad \text{for } r \neq 0. \tag{4.53}$$

Since the fields are static the conservation law eq.(4.3) continues to hold true, as there is no motion of charges and for any closed surface containing the spherical conductor we have

$$\oint_S d\vec{S}.\vec{P} = 0. \tag{4.54}$$

But *nothing* is in motion! In view of these results we must investigate whether the existence of superluminal UPWs solutions of ME is compatible or not with the Principle of Relativity. We analyze this question in detail in the next Section.

To end this Section we recall that in Section 2.19 of his book [14] Stratton presents a discussion of the Poynting vector and energy transfer which essentially agrees with the view presented above. Indeed he finished that Section with the words: "By this standard there is every reason to retain the Poynting-Heaviside viewpoint until a clash with new experimental evidence shall call for its revision."

5. SUPERLUMINAL SOLUTIONS OF MAXWELL EQUATIONS AND THE PRINCIPLE OF RELATIVITY

In [6] it was shown that it seems possible with present technology to launch in free space finite aperture approximations to the superluminal electromagnetic waves (SEXWs). We show in the following that the physical existence of SEXWs implies a breakdown of the Principle of Relativity (PR). Since this is a fundamental issue, with implications for all branches of theoretical physics, we will examine the problem with great care. In Section 5.1 we give a rigorous mathematical definition of the PR and in Section 5.2 we present the proof of the above statement.

5.1. Mathematical formulation of the Principle of Relativity and its physical meaning

In Section 2 we defined Minkowski spacetime as the triple $\langle M, g, D\rangle$, where $M \simeq I\!R^4$, g is a Lorentzian metric and D is the Levi-Civita connection of g. Consider now G_M, the group of all diffeomorphisms of M, called the manifold mapping group. Let $\mathbf{T}$ be a geometrical object defined in $A \subseteq M$. The diffeomorphism $h \in G_M$ induces a deforming mapping $h_* : \mathbf{T} \to h_*\mathbf{T} = \mathbf{T}$ such that:

(i) If $f : M \supseteq A \to I\!R$, then $h_* f = f \circ h^{-1} : h(A) \to I\!R$.

(ii) If $\mathbf{T} \in \sec T^{(r,s)}(A) \subset \sec T(M)$, where $T^{(r,s)}(A)$ is the sub-bundle of tensors of type (r, s) of the tensor bundle $T(M)$, then

$$(h_*\mathbf{T})_{h_e}(h_*\omega_1, \ldots, h_*\omega_r, h_*X_1, \ldots, h_*X_s) = \mathbf{T}_e(\omega_1, \ldots, \omega_r, X_1, \ldots, X_s) \quad (5.1)$$

$\forall X_i \in T_eA$, $i = 1, \ldots, s$, $\forall \omega_j \in T^*_eA$, $j = 1, \ldots, r$, $\forall e \in A$.

(iii) If D is the Levi-Civita connection and $X, Y \in \sec TM$, then

$$(h_* D_{h_*x} h_* Y)_{he} h_{*f} = (D_x Y)_e f \quad \forall e \in M\,. \quad (5.2)$$

If $\{f_\mu = \partial/\partial x^\mu\}$ is a coordinate basis for TA and $\{\theta^\mu = \mathrm{d}x^\mu\}$ is the corresponding dual basis for T^*A and if

$$\mathbf{T} = T^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s} \theta^{\nu_1} \otimes \ldots \otimes \theta^{\nu_s} \otimes f_{\mu_1} \otimes \ldots \otimes f_{\mu_r}, \quad (5.3)$$

then

$$h_*\mathbf{T} = [T^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s} \circ h^{-1}] h_*\theta^{\nu_1} \otimes \ldots \otimes h_*\theta^{\nu_r} \otimes h_* f_{\mu_1} \otimes \ldots \otimes h_* f_{\mu_s} \quad (5.4)$$

Suppose now that A and $h(A)$ can be covered by the local chart (U, η) of the maximal atlas of M, and $A \subseteq U, h(A) \subseteq U$. Let $\langle x^\mu\rangle$ be the coordinate functions associated with (U, η). The mapping

$$x^{'\mu} = x^\mu \circ h^{-1} \;:\; h(U) \to I\!R \quad (5.5)$$

defines a coordinate transformation $\langle x^\mu\rangle \mapsto \langle x^{'\mu}\rangle$ if $h(U) \supseteq A \cup h(A)$. Indeed $\langle x^{'\mu}\rangle$ are the coordinate functions associated with the local chart (V, φ) where $h(U) \subseteq V$ and

$U \cap V \neq \phi$. Now, since it is well known that under the above conditions $h_* \partial/\partial x^\mu \equiv \partial/\partial x^{'\mu}$ and $h_* \mathrm{d}x^\mu \equiv \mathrm{d}x^{'\mu}$, eqs.(5.1), (5.3) and (5.4) imply that

$$(h_* \mathbf{T})_{\langle x^{'\mu} \rangle}(he) = \mathbf{T}_{\langle x^\mu \rangle}(e)\,, \tag{5.6}$$

where $\mathbf{T}_{\langle x^\mu \rangle}(e)$ means the components of $\mathbf{T}$ in the chart $\langle x^\mu \rangle$ at the event $e \in M$, *i.e.* $\mathbf{T}_{\langle x^\mu \rangle}(e) = T^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s}(x^\mu(e))$ and where $\overline{T}^{'\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s}(x^{'\mu}(he))$ are the components of $\overline{\mathbf{T}} = h_* \mathbf{T}$ in the basis $\{h_* \partial/\partial x^\mu = \partial/\partial x^{'\mu}\}$, $\{h_* \mathrm{d}x^\mu = \mathrm{d}x^{'\mu}\}$, at the point $h(e)$. Then eq.(5.6) reads

$$\overline{T}^{'\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s}(x^{'\mu}(he)) = T^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s}(x^\mu(e)) \tag{5.7}$$

or using eq.(5.5)

$$\overline{T}^{'\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s}(x^{'\mu}(e)) = (\Lambda^{-1})^{\mu_1}_{\alpha_1} \ldots \Lambda^{\beta_s}_{\nu_s} T^{'\alpha_1 \ldots \alpha_r}_{\beta_1 \ldots \beta_s}(x^{'\mu}(h^{-1}e)) \tag{5.8}$$

where $\Lambda^\mu_\alpha = \partial x^{'\mu}/\partial x^\alpha$, etc.

In Section 2 we already introduced the concept of inertial reference frames $I \in \sec TU$, $U \subseteq M$ by

$$g(I, I) = 1 \quad \text{and} \quad DI = 0 \tag{5.9}$$

A general frame Z satisfies $g(Z, Z) = 1$, with $DZ \neq 0$. If $\alpha = g(Z,\ ) \in \sec T^*U$, it holds

$$(D\alpha)_e = a_e \otimes \alpha_e + \sigma_e + \omega_e + \frac{1}{3}\theta_e h_e, e \in U \subseteq M, \tag{5.10}$$

where $a = g(\mathrm{A},\ )$, $\mathrm{A} = D_Z Z$ is the acceleration and where ω_e is the rotation tensor, σ_e is the shear tensor, θ_e is the expansion and $h_e = g_{|H_e}$ where

$$T_e M = [Z_e] \oplus [H_e]\,. \tag{5.11}$$

H_e is the rest space of an *instantaneous observer* at e, *i.e.* the pair (e, Z_e). Also $h_e(X, Y) = g_e(p(pX, pY), \forall X, Y \in T_e M$ and $p : T_e M \to H_e$. (For the explicit form of ω, σ, θ, see [11, 44]). From eqs.(5.9) and (5.10) we see that an inertial reference frame has no acceleration, has no rotation, no shear and no expansion.

We introduced also in Section 2 the concept of a (nacs/I). A (nacs/I) $\langle x^\mu \rangle$ is said to be in the Lorentz gauge if $x^\mu, \mu = 0, 1, 2, 3$ are the usual Lorentz coordinates and $I = \partial/\partial x^0 \in \sec TM$. We recall that it is a theorem that putting $I = e_0 = \partial/\partial x^0$, there exist three other fields $e_i \in \sec TM$, such that $g(e_i, e_i) = -1,\ i = 1, 2, 3$, and $e_i = \partial/\partial x^i$.

Now, let $\langle x^\mu \rangle$ be Lorentz coordinate functions as above. We say that $\ell \in G_M$ is a *Lorentz mapping* if and only if

$$x^{'\mu}(e) = \Lambda^\mu_\nu x^\mu(e), \tag{5.12}$$

where $\Lambda^\mu_\nu \in \mathcal{L}^\uparrow_+$ is a Lorentz transformation. For abuse of notation we denote the subset $\{\ell\}$ of G_M such that eq.(5.12) holds true also by $\mathcal{L}^\uparrow_+ \subset G_M$.

When $\langle x^\mu \rangle$ are Lorentz coordinate functions, $\langle x^{'\mu} \rangle$ are also Lorentz coordinate functions. In this case we denote

$$e_\mu = \partial/\partial x^\mu,\ e'_\mu = \partial/\partial x^{'\mu},\ \gamma_\mu = \mathrm{d}x^\mu,\ \gamma'_\mu = \mathrm{d}x^{'\mu}\ ; \tag{5.13}$$

when $\ell \in \mathcal{L}^\uparrow_+ \subset G_M$ we say that $\ell_* \mathbf{T}$ is the *Lorentz deformed version* of $\mathbf{T}$.

Let $h \in G_M$. If for a geometrical object $\mathbf{T}$ we have

$$h_* \mathbf{T} = \mathbf{T}, \tag{5.14}$$

then h is said to be a symmetry of $\mathbf{T}$ and the set of all $h \in G_M$ such that eq.(5.13) holds is said to be the symmetry group of $\mathbf{T}$. We can immediately verify that for $\ell \in \mathcal{L}_+^\uparrow \subset G_M$

$$\ell_* g = g,\ \ell_* D = D, \tag{5.15}$$

i.e., the special restricted orthochronous Lorentz group $\mathcal{L}_+^\uparrow$ is a symmetry group of g and D.

In [12] we maintain that a physical theory τ is characterized by:

(i) the theory of a certain "species of structure" in the sense of Boubarki [45];

(ii) its physical interpretation;

(iii) its present meaning and present applications.

We recall that in the mathematical exposition of a given physical theory τ, the postulates or basic axioms are presented as definitions. Such definitions mean that the physical phenomena described by τ behave in a certain way. Then, the definitions require more motivation than the pure mathematical definitions. We call coordinative definitions the physical definitions, a term introduced by Reichenbach [46]. It is necessary also to make clear that completely convincing and genuine motivations for the coordinative definitions cannot be given, since they refer to nature as a whole and to the physical theory as a whole.

The theoretical approach to physics behind (i), (ii) and (iii) above is then to admit the mathematical concepts of the "species of structure" defining τ as primitives, and define coordinatively the observation entities from them. Reichenbach assumes that "*physical knowledge* is characterized by the fact that concepts are not only defined by other concepts, but are also coordinated to real objects". However, in our approach, each physical theory, when characterized as a species of structure, contains some implicit geometric objects, like some of the reference frame fields defined above, that cannot in general be coordinated to real objects. Indeed it would be an absurd to suppose that all the infinity of IRF that exist in M must have a material support.

We define a *spacetime* theory as a theory of a species of structure such that, if Mod τ is the class of models of τ, then each $\Upsilon \in$ Mod τ contains a substructure called spacetime (ST). More precisely, we have

$$\Upsilon = (\mathrm{ST}, \mathbf{T}_1, \dots, \mathbf{T}_m\}\ , \tag{5.16}$$

where ST can be a very general structure [12]. For what follows we suppose that $\mathrm{ST} = \mathcal{M} = (M, g, D)$ *i.e.*, that ST is Minkowski spacetime. The $\mathbf{T}_i, i = 1, \dots, m$ are (explicit) geometrical objects defined in $U \subseteq M$ characterizing the physical fields and particle trajectories that cannot be geometrized in Υ. Here, to be geometrizable means to be a metric field or a connection on M or objects derived from these concepts, as *e.g.*, the Riemann tensor or the torsion tensor in more general theories. The reference frame fields will be called the *implicit* geometrical objects of τ, since they are mathematical objects that do not necessarily correspond to properties of a physical system described by τ.

Now, with the Clifford bundle formalism we can formulate in $\mathcal{C}\ell(M)$ all modern physical theories (see Section 2) including Einstein's gravitational theory [1]. We introduce now the Lorentz-Maxwell electrodynamics (LME) in $\mathcal{C}\ell(M)$ as a theory of a species of structure. We say that LME has as model

$$\Upsilon_{LME} = \langle M, g, D, F, J, \{\varphi_i, m_i, e_i\}\rangle \tag{5.17}$$

where (M, g, D) is Minkowski spacetime, $\{\varphi_i, m_i, e_i\}$, $i = 1, 2, \ldots, N$ is the set of all charged particles, m_i and e_i being the masses and charges of the particles and $\varphi_i : I\!R \supset I \to M$ being the world lines of the particles characterized by the fact that if $\varphi_{i*} \in \sec TM$ is the velocity vector, then $\check{\varphi}_i = g(\varphi_{i*},\ ) \in \sec \Lambda^1(M) \subset \sec \mathcal{C}\ell(M)$ and $\check{\varphi}_i . \check{\varphi}_i = 1$. $F \in \sec \Lambda^2(M) \subset \sec \mathcal{C}\ell(M)$ is the electromagnetic field and $J \in \sec \Lambda^1(M) \subset \sec \mathcal{C}\ell(M)$ is the current density. The proper axioms of the theory are

$$\begin{gathered} \partial F = J\,, \\ m_i D_{\varphi_{i*}} \check{\varphi}_i = e_i \check{\varphi}_i \cdot F\,. \end{gathered} \tag{5.18}$$

From a mathematical point of view it is a trivial result that τ_{LME} has the following property: If $h \in G_M$ and if the set of eqs.(5.16) has a solution $\langle F, J, (\varphi_i, m_i, e_i)\rangle$ in $U \subseteq M$ then $\langle h_*F, h_*J, (h_*\varphi_i, m_i, e_i)\rangle$ is also a solution of eqs.(5.16) in $h(U)$. Since the result is true for any $h \in G_M$ it is true for $\ell \in \mathcal{L}_+^\uparrow \subset G_M$, *i.e.* for any Lorentz mapping.

We must now make it clear that $\langle F, J, \{\varphi_i, m_i, e_i\}\rangle$ which is a solution of eq.(5.16) in U can be obtained only by imposing *mathematical boundary conditions* which we denote by BU. The solution will be realizable in nature if and only if the mathematical boundary conditions can be physically realizable. This is indeed a nontrivial point [12] for in particular it says to us that even if $\langle h_*F, h_*J, \{h_*\varphi_i, m_i, e_i\}\rangle$ can be a solution of eqs.(5.16) with mathematical boundary conditions $Bh(U)$, it may happen that $Bh(U)$ cannot be physically realizable in nature. The following statement, denoted PR_1, is usually presented [12] as the Principle of (Special) Relativity in active form:

PR_1: Let $\ell \in \mathcal{L}_+^\uparrow \subset G_M$. If for a physical theory τ we have $\Upsilon \in \mathrm{Mod}\,\tau$, where $\Upsilon = \langle M, g, D, \mathbf{T}_1, \ldots, \mathbf{T}_m\rangle$ is a possible physical phenomenon, then $\ell_*\Upsilon = \langle M, g, D, l_*\mathbf{T}_1, \ldots, l_*\mathbf{T}_m\rangle$ is also a possible physical phenomenon.

It is clear that *hidden* in PR_1 is the assumption that the boundary conditions that determine $\ell_*\Upsilon$ are physically realizable. Before we continue we introduce the statement denoted PR_2 known as the Principle of (Special) Relativity in passive form [12].

PR_2: All inertial reference frames are physically equivalent or indistinguishable.

We now give a precise mathematical meaning to the above statement.

Let τ be a spacetime and let $\mathrm{ST} = \langle M, g, D\rangle$ be a substructure of $\mathrm{Mod}\ \tau$ representing spacetime. Let $I \in \sec TU$ and $I' \in \sec TV, U, V \subseteq M$ be two inertial reference frames. Let (U, η) and (V, φ) be two Lorentz charts of the maximal atlas of M that are naturally adapted respectively to I and I'. For $\langle x^\mu\rangle$ and $\langle x^{'\mu}\rangle$ the coordinate functions associated with (U, η) and (V, φ) we have $I = \partial/\partial x^0, I' = \partial/\partial x^{'0}$.

Definition. Two inertial reference frames I and I' as above are said to be physically equivalent according to τ if and only if the following conditions are satisfied:

(i) $G_M \supset \mathcal{L}_+^\uparrow \ni \ell : U \to \ell(U) \subseteq V\,,\ x^{'\mu} = x^\mu \circ \ell^{-1} \Rightarrow I' = \ell_* I$

When $\Upsilon \in \mathrm{Mod}\ \tau$, $\Upsilon = \langle M, g, D, \mathbf{T}_1, \ldots \mathbf{T}_m\rangle$, is such that g and D are defined over all M and $\mathbf{T}_i \in \sec \mathcal{C}\ell(U) \subset \sec \mathcal{C}\ell(M)$, $i = 1, \ldots, m$, calling $o = \langle g, D, \mathbf{T}_1, \ldots \mathbf{T}_m\rangle$,

o solves a set of differential equations in $\eta(U) \subset I\!R^4$ with a given set of boundary conditions denoted $b^{o\langle x^\mu\rangle}$, which we write as

$$D^\alpha_{\langle x^\mu\rangle}(o_{\langle x^\mu\rangle})_e = 0 \ ; \ b^{o\langle x^\mu\rangle} \ ; \ e \in U \tag{5.19}$$

and we must have:

(ii) If $\Upsilon \in \mathrm{Mod}\ \tau \Leftrightarrow \ell_*\Upsilon \in \mathrm{Mod}\ \tau$, then necessarily

$$\ell_*\Upsilon = \langle M, g, D, \ell_*\mathbf{T}_1, \ldots \ell_*\mathbf{T}_m\rangle \tag{5.20}$$

is defined in $\ell(U) \subseteq V$ and calling $\ell_* o \equiv \{g, D, \ell_*\mathbf{T}_1, \ldots, \ell_*\mathbf{T}_m\}$ we must have

$$D^\alpha_{\langle x'^\mu\rangle}(\ell_* o_{\langle x'^\mu\rangle})_{|\ell e} = 0 \ ; \ b^{\ell_* o\langle x'^\mu\rangle} \quad \ell e \in \ell(U) \subseteq V. \tag{5.21}$$

The system of differential equations (5.19) must have the same functional form as the system of differential equations (5.17) and $b^{\ell_* o\langle x'^\mu\rangle}$ must be relative to $\langle x'^\mu\rangle$ the same as $b^{o\langle x^\mu\rangle}$ is relative to $\langle x^\mu\rangle$ and if $b^{o\langle x^\mu\rangle}$ is physically realizable then $b^{\ell_* o\langle x'^\mu\rangle}$ must also be physically realizable. We say under these conditions that $I \sim I'$ and that $\ell_* o$ is the Lorentz version of the phenomenon described by o.

Since in the above definition $\ell_*\Upsilon = \langle M, g, D, \ell_* T_1, \ldots, \ell_* T_m\rangle$, it follows that when $I \sim I'$, then $\ell_* g = g, \ell_* D = D$ (as we already know) and this means that the spacetime structure does not give a preferred status to I or I' according to τ.

5.2. Proof that the existence of SEXWs implies a breakdown of PR$_1$ and PR$_2$

We are now able to prove the statement presented in the beginning of this Section, that the existence of SEXWs implies a breakdown of the Principle of Relativity in both its active (PR_1) and passive (PR_2) versions.

Let $\ell \in \mathcal{L}_+^\uparrow \subset G_M$ and let F, $\overline{F} \in \sec\Lambda^2(M) \subset \sec\mathcal{C}\ell(M)$, $\overline{F} = \ell_* F$. Let $\overline{F} = \ell_* F = R\check{F}R^{-1}$, where $\check{F}_e = (1/2)F_{\mu\nu}(x^\mu(\ell^{-1}e))\gamma^\mu\gamma^\nu$ and where $R \in \sec \mathrm{Spin}_+(1,3) \subset \sec\mathcal{C}\ell(M)$ is a Lorentz mapping, such that $\gamma'^\mu = R\gamma^\mu R^{-1} = \Lambda^\mu_\alpha\gamma^\alpha$, $\Lambda^\mu_\alpha \in \mathcal{L}_+^\uparrow$ and let $\langle x^\mu\rangle$ and $\langle x'^\mu\rangle$ be Lorentz coordinate functions as before such that $\gamma^\mu = \mathrm{d}x^\mu$, $\gamma'^\mu = \mathrm{d}x'^\mu$ and $x'^\mu = x^\mu \circ \ell^{-1}$. We write

$$F_e = \frac{1}{2}F_{\mu\nu}(x^\mu(e))\gamma^\mu\gamma^\nu \tag{5.22a}$$

$$F_e = \frac{1}{2}F'_{\mu\nu}(x'^\mu(e))\gamma'^\mu\gamma'^\nu \tag{5.22b}$$

$$\overline{F}_e = \frac{1}{2}\overline{F}_{\mu\nu}(x^\mu(e))\gamma^\mu\gamma^\nu \tag{5.23a}$$

$$\overline{F}_e = \frac{1}{2}\overline{F}'_{\mu\nu}(x'^\mu(e))\gamma'^\mu\gamma'^\nu. \tag{5.23b}$$

From (5.22a) and (5.22b) we get that

$$F'_{\alpha\beta}(x'^\mu(e)) = (\Lambda^{-1})^\mu_\alpha(\Lambda^{-1})^\nu_\beta F_{\mu\nu}(x^\mu(e)). \tag{5.24}$$

From (5.23a) and (5.23b) we also get

$$\overline{F}_{\alpha\beta}(x^\mu(e)) = \Lambda^\mu_\alpha \Lambda^\nu_\beta F_{\mu\nu}(x^\mu(\ell^{-1}e))\,. \tag{5.25}$$

Now, suppose that F is a superluminal solution of Maxwell equation, in particular a SEXW as discussed in Section 3. Suppose that F has been produced in the inertial frame I with $\langle x^\mu \rangle$ as (nacs/I), with the physical device described in Section 3. F is then traveling with speed $c_1 = 1/\cos\eta$ in the negative z-direction and being generated in the plane $z = 0$, will travel to the future in spacetime, according to the observers in I. Now, there exists $\ell \in \mathcal{L}_+^\uparrow$ such that $\ell_* F = \overline{F} = RFR^{-1}$ will be a solution of Maxwell equations and such that if the velocity 1-form of F is $v_F = (c_1^2 - 1)^{-1/2}(1, 0, 0, -c_1)$, then the velocity 1-form of $\overline{F}$ is $v_{\overline{F}} = (c_1'^2 - 1)^{-1/2}(-1, 0, 0, -c_1')$, with $c_1' > 1$, i.e, $v_{\overline{F}}$ is pointing to the past. As its is well known $\overline{F}$ carries negative energy density according to the observers in the I frame.

We then arrive at the conclusion that to assume the validity of PR_1 is to assume the physical possibility of sending to the past waves carrying negative energy. This seems to the authors an impossible task, and the reason is that there do not exist physically realizable boundary conditions that permit the observers in I to launch $\overline{F}$ in spacetime and such that it travels to its own past.

We now show that there is also a breakdown of PR_2, *i.e.*, that it is not true that all inertial frames are physically equivalent. Suppose we have two inertial frames I and I' as above i.e, $I = \partial/\partial x^0$, $I' = \partial/\partial x'^0$. Suppose that F is a SEXW which can be launched in I with velocity 1-form as above and suppose $\overline{F}$ is a SEXW built in I' at the plane $z' = 0$ and with velocity 1-form relative to $\langle x'^\mu \rangle$ given by $v_{\overline{F}} = v'^\mu \gamma'_\mu$ and

$$v_{\overline{F}} = \left(\frac{1}{\sqrt{c_1^2 - 1}}, 0, 0, -\frac{c_1}{\sqrt{c_1^2 - 1}} \right) \tag{5.26}$$

If F and $\overline{F}$ are related as above we see that $\overline{F}$, which has positive energy and is traveling to the future according to I', *can be sent* to the past of the observers at rest in the I frame. Obviously this is impossible and we conclude that $\overline{F}$ is not a physically realizable phenomenon in nature. It cannot be realized in I' but F can be realized in I. It follows that PR_2 does not hold.

If the set of inertial reference frames are not equivalent then there must exist a fundamental reference frame. Let $I \in \sec TM$ be the fundamental frame. If I' is moving with speed V relative to I, *i.e.*

$$I' = \frac{1}{\sqrt{1 - V^2}} \frac{\partial}{\partial t} - \frac{V}{\sqrt{1 - V^2}} \frac{\partial}{\partial z}\,, \tag{5.27}$$

then, if observers in I' are equipped with a generator of SEXWs and if they prepare their apparatus in order to send SEXWs with different velocity 1-forms in all possible directions in spacetime, they will find a particular velocity 1-form in a given spacetime direction in which the device stops working. A simple calculation yields then, for the observers in I', the value of V!

In [47] Recami argued that the Principle of Relativity continues to hold true even if superluminal phenomena exist in nature. In this theory of tachyons there exists, of course, a situation completely analogous to the one described above (called the Tolman-Regge paradox), and according to Recami's view PR_2 is valid because I' must interpret $\overline{F}$ a being an anti-SEXW carrying positive energy and going into the future according to

him. In his theory of tachyons Recami was able to show that the dynamics of tachyons implies that no detector *at rest* in I can detect a tachyon (the same would be valid for a SEXW like $\overline{F}$) sent by I' with velocity 1-form given by eq.(5.26). Thus he claimed that PR_2 is true. At first sight the argument seems good, but it is at least incomplete. Indeed, a detector in I does not need to be at rest in I. We can imagine a detector in periodic motion in I which can absorb the $\overline{F}$ wave generated by I' if this was indeed possible. It is enough for the detector to have relative to I the speed V of the I' frame in the appropriate direction at the moment of absorption. This simple argument shows that there is no salvation for PR_2 (and for PR_1) if superluminal phenomena exist in nature. Our argumentation is endorsed by Barashenkov and Yur'iev [48].

The attentive reader at this point probably has the following question in his/her mind: How could the authors start with Minkowski spacetime, with equations carrying the Lorentz symmetry and yet arrive at the conclusion that PR_1 and PR_2 do not hold? The reason is that the Lorentzian structure of $\langle M, g, D\rangle$ can be seen to exist directly from the Newtonian spacetime structure as proved in [49]. In this paper, Rodrigues and collaborators show that even if $\mathcal{L}_+^\uparrow$ is not a symmetry group of Newtonian dynamics it is a symmetry group of the only possible coherent formulation of Lorentz-Maxwell electrodynamic theory compatible with experimental results that is possible to formulate in the Newtonian spacetime.[12]

We finish calling to the reader's attention that there are some experiments reported in the literature which suggest also a breakdown of PR_2 for the roto-translational motion of solid bodies. A discussion and references can be found in [13]. A coherent spacetime model which can accommodate superluminal phenomena has been recently proposed by Matolcsi and Rodrigues [50].

6. CONCLUSIONS

In this paper we presented a unified theory showing that the homogeneous wave equation, the Klein-Gordon equation, Maxwell equations and the Dirac and Weyl equations have solutions with the form of undistorted progressive waves (UPWs) of arbitrary speeds $0 \leq v < \infty$. We exhibit also some subluminal and superluminal solutions of Maxwell equations. We showed that subluminal solution can in principle be used to model purely electromagnetic particles.

The possible existence of superluminal electromagnetic waves implies in a breakdown of the Principle of Relativity. It is important to recall here that exact Lorentz symmetry can be preserved in an abstract mathematical level through the ingenious construction of Santilli's isominkowskian spaces (see [51-55]). Santilli's theory is important, *e.g.* for situations involving the hadronic medium, where superluminal velocities can occur. We observe that besides the fundamental theoretical implications, the practical implications of the existence of UPWs solutions of the main field equations of theoretical physics (and their finite aperture realizations) are very important. This practical importance ranges from applications in ultrasound medical imaging to the project of electromagnetic bullets and new communication devices [56]. Also, we would like to conjecture that the existence of subluminal and superluminal solutions of the Weyl equation may be important to solve some of the mysteries associated with neutrinos. Indeed, if neutrinos can be produced in subluminal or superluminal modes (see [57, 58] for some experimental evidence concerning superluminal neutrinos) they can eventually escape detection on earth after leaving the sun. Moreover, for neutrinos in a subluminal or

[12] We recall that Maxwell equations have, as is well known, many symmetry groups besides $\mathcal{L}_+^\uparrow$.

superluminal mode it would be possible to define a kind of "effective mass". Recently some cosmological evidences that neutrinos have a non-vanishing mass have been discussed, *e.g.*, by Primack et al [59]. One such "effective mass" could be responsible for those cosmological evidences, and in such a way we can still have a left-handed neutrino since it would satisfy the Weyl equation. We shall discuss more this issue in another publication.

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